Spectral Theory

Homework 6

1. Prove that the norm $\|\cdot\|_p$ on l^p, $p \neq 2$, is not induced by an inner product. (*Hint:* Prove that for $x = (1, 1, 0, \dots) \in l^p$ and $y = (1, -1, 0, \dots) \in l^p$ the parallelogram law fails.)

2. Prove that the norm $\|\cdot\|_p$, $p \neq 2$ on $C([0,1])$ is not induced by an inner product. (*Hint:* Prove that for functions $f(t) = 1/2 - t$ and

$$g(t) = \begin{cases} 1/2 - t & \text{if } 0 \le t \le 1/2 \ , \\ t - 1/2 & \text{if } 1/2 < t \le 1 \ , \end{cases}$$

the parallelogram law fails).

3. Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal set in an inner product space $\mathcal{H}$. Prove that

$$\sum_{n=1}^{\infty} |(x, e_n)(y, e_n)| \le \|x\| \|y\|, \quad \forall x, y \in \mathcal{H}.$$

4. Show that $A^{\perp\perp} = \overline{\text{span} A}$ for any subset of a Hilbert space.

5. Let M and N be closed subspaces of a Hilbert space. Show that $(M+N)^{\perp} = M^{\perp} \cap N^{\perp}, \quad (M \cap N)^{\perp} = \overline{(M^{\perp} + N^{\perp})}$.

6. Show that $M := \{x = (x_n) \in l^2 : \ x_{2n} = 0, \ \forall n \in \mathbb{N}\}$ is a closed subspace of l^2. Find $M^{\perp}$.

7. Show that vectors $x_1, \dots, x_N$ in an inner product space $\mathcal{H}$ are linearly independent iff their *Gram matrix* $(a_{jk})_{j,k=1}^N = ((x_k, x_j))_{j,k=1}^N$ is nonsingular, i.e. iff the corresponding *Gram determinant* $\det((x_k, x_j))$ does not equal zero. Take an arbitrary $x \in \mathcal{H}$ and set $b_j = (x, x_j)$. Show that, whether or not x_j are linearly independent, the system of equations

$$\sum_{k=1}^{N} a_{jk} c_k = b_j, \quad j = 1, \dots, N,$$

is solvable and that for any solution $(c_1, \dots, c_N)$ the vector $\sum_{j=1}^N c_j x_j$ is the nearest to x point of $\text{span}\{x_1, \dots, x_N\}$.