SPECTRAL THEORY

HOMEWORK 3

1. Let X be a Banach space and $A, B \in \mathcal{B}(X)$.
(a) Show that if $I - AB$ is invertible, then $I - BA$ is also invertible. [*Hint:* consider $B(I - AB)^{-1}A + I$.]
(b) Prove that if $\lambda \in \sigma(AB)$ and $\lambda \neq 0$, then $\lambda \in \sigma(BA)$.
(c) Give an example of operators A and B such that $0 \in \sigma(AB)$ but $0 \notin \sigma(BA)$.
(d) Show that $\sigma(AB) \bigcup \{0\} = \sigma(BA) \bigcup \{0\}$.
(e) Prove that $r(AB) = r(BA)$.

2. Let X be a Banach space and let operators $A, B \in \mathcal{B}(X)$ commute: $AB = BA$. Prove that $r(A + B) \leq r(A) + r(B)$.

3. Let $k \in C([0,1] \times [0,1])$ be a given function. Consider the operator $B \in \mathcal{B}(C([0,1]))$ defined by the formula

$$(Bu)(s) = \int_0^s k(s,t)u(t)dt.$$

Find the spectral radius of B. What is the spectrum of B? [*Hint:* prove by induction that

$$|(B^n u)(s)| \leq \frac{M^n}{n!} s^n \|u\|_\infty, \quad \forall n \in \mathbb{N},$$

for some constant $M > 0$.]