SPECTRAL THEORY

HOMEWORK 3

1. Let $X$ be a Banach space and $A, B \in \mathcal{B}(X)$.
   (a) Show that if $I - AB$ is invertible, then $I - BA$ is also invertible. [*Hint: consider $B(I - AB)^{-1}A + I.$]*
   (b) Prove that if $\lambda \in \sigma(AB)$ and $\lambda \neq 0$, then $\lambda \in \sigma(BA)$.
   (c) Give an example of operators $A$ and $B$ such that $0 \in \sigma(AB)$ but $0 \not\in \sigma(BA)$.
   (d) Show that $\sigma(AB) \cup \{0\} = \sigma(BA) \cup \{0\}$.
   (e) Prove that $r(AB) = r(BA)$.

2. Let $X$ be a Banach space and let operators $A, B \in \mathcal{B}(X)$ commute: $AB = BA$. Prove that $r(A + B) \leq r(A) + r(B)$.

3. Let $k \in C([0,1] \times [0,1])$ be a given function. Consider the operator $B \in \mathcal{B}(C([0,1]))$ defined by the formula

\[
(Bu)(s) = \int_0^s k(s,t)u(t)dt.
\]

Find the spectral radius of $B$. What is the spectrum of $B$? [*Hint: prove by induction that

\[
|(B^n u)(s)| \leq \frac{M^n}{n!} s^n \|u\|_{\infty}, \quad \forall n \in \mathbb{N},
\]

for some constant $M > 0.$]