SPECTRAL THEORY

HOMEWORK 1

1. Prove that $\mathcal{B}(\mathbb{F}, Y)$ is not a Banach space if Y is not complete. [*Hint:* take a Cauchy sequence (y_n) in Y which does not converge and consider the sequence of operators (B_n),

$$B_n\lambda := \lambda y_n, \ \forall \lambda \in \mathbb{F}.]$$

2. Give an example of a bounded linear operator A such that $\mathrm{Ran}(A)$ is not closed. [*Hint:* consider the imbedding $X \to Y$, where X is the space $C([0,1])$ equipped with the norm $\|\cdot\|_\infty$ and $Y = L_p([0,1])$ is the completion of the normed space $(C([0,1]), \|\cdot\|_p)$, $1 \le p < \infty$.]

3. Give an example of a normed space and an absolutely convergent series in it, which is not convergent.

4. Let X be the Banach space $C([0,1])$ and Y be the space of all continuously differentiable functions on $[0,1]$ which equal 0 at 0. Both of the spaces are equipped with the norm $\|\cdot\|_\infty$. Show that the linear operator $B : X \to Y$,

$$(Bf)(t) := \int_0^t f(\tau)d\tau,$$

is bounded, one–to–one and onto, but the inverse operator $B^{-1} : Y \to X$ is not bounded. Compare this with the Banach theorem (bounded inverse theorem).

5. Denote by $C^2([0,1])$ the space of twice continuously differentiable functions on the interval $[0,1]$ equipped with the norm

$$\|u\| := \max_{0\le s\le 1} |u(s)| + \max_{0\le s\le 1} |u'(s)| + \max_{0\le s\le 1} |u''(s)|.$$

Let X denote the subspace of $C^2([0,1])$ containing functions satisfying boundary conditions $u(0) = u(1) = 0$. Prove that the operator $A = -\frac{d^2}{ds^2}$ is a bounded operator acting from X to $C([0,1])$.

6. Show that the operator A from the previous question has a bounded inverse $A^{-1} : C([0,1]) \to X$ and construct it effectively.