OPERATOR THEORY

Solution VIII

1. Let \( g \in C([0,1]) \) be a given function. Consider the operator \( A \in B(L_2([0,1])) \) defined by the formula
\[
(Au)(s) = g(s)u(s), \quad s \in [0,1].
\]
Find the operator \( A^* \). Under what condition on \( g \) is the operator \( A \) self-adjoint?

**Solution**
\[
(Au,v) = \int_0^1 g(s)u(s)\overline{v(s)}\,ds = \int_0^1 u(s)g(s)\overline{v(s)}\,ds = (u,A^*v),
\]
where
\[
(A^*v)(s) = g(s)v(s), \quad s \in [0,1].
\]
It is clear that \( A \) is self-adjoint iff \( g = \overline{g} \), i.e. iff \( g \) is real-valued.

2. Let \( k \in C([0,1] \times [0,1]) \) be a given function. Consider the operator \( B \in B(L_2([0,1])) \) defined by the formula
\[
(Bu)(s) = \int_0^1 k(s,t)u(t)\,dt, \quad s \in [0,1].
\]
Find the operator \( B^* \). Under what condition on \( k \) is the operator \( B \) self-adjoint?

**Solution**
\[
(Bu,v) = \int_0^1 \left( \int_0^1 k(s,t)u(t)\,dt \right) \overline{v(s)}\,ds = \int_0^1 u(t) \int_0^1 k(s,t)\overline{v(s)}\,ds \,dt = (u,B^*v),
\]
where
\[
(B^*v)(t) = \int_0^1 k(s,t)v(s)\,ds, \quad t \in [0,1],
\]
\[(B^*v)(s) = \int_0^1 k(t, s)v(t)dt, \ s \in [0, 1].\]

It is clear that \(B\) is self-adjoint iff \(k(s, t) = k(t, s), \forall s, t \in [0, 1]\).

3. Let \(B\) be defined by
\[(Bf)(t) = tf(1 - t^3), \ \forall f \in L_2([0, 1]), \ \forall t \in [0, 1].\]

Prove that \(B \in \mathcal{B}(L_2([0, 1]))\) and find \(B^*\), \(BB^*\) and \(B^*B\).

**Solution**

Using the change of variable \(\tau = 1 - t^3\) we obtain
\[
\|Bf\|^2 = \int_0^1 t^2|f(1 - t^3)|^2dt = -\frac{1}{3} \int_1^0 |f(\tau)|^2d\tau =
\frac{1}{3} \int_0^1 |f(\tau)|^2d\tau = \frac{1}{3} \|f\|^2, \ f \in L_2([0, 1]).
\]

Hence \(B \in \mathcal{B}(L_2([0, 1]))\) and \(\|B\| = 1/\sqrt{3}\).

Further,
\[
(Bf, g) = \int_0^1 tf(1 - t^3)g(t)dt = \frac{1}{3} \int_0^1 \frac{1}{\sqrt{1 - \tau}}f(\tau)g(\sqrt{1 - \tau})d\tau =
\frac{1}{3} \int_0^1 f(\tau)\frac{1}{\sqrt{1 - \tau}}g(\sqrt{1 - \tau})d\tau = (f, B^*g),
\]

where
\[
(B^*g)(\tau) = \frac{1}{3\sqrt{1 - \tau}}g(\sqrt{1 - \tau}), \ \tau \in [0, 1].
\]

Consequently
\[
(BB^*g)(t) = \frac{1}{3\sqrt{1 - (1 - t^3)}}g(\sqrt{1 - (1 - t^3)}) = \frac{1}{3}g(t), \ t \in [0, 1],
\]
\[
(B^*Bf)(\tau) = \frac{1}{3\sqrt{1 - \tau}}\sqrt{1 - \tau}f(1 - (\sqrt{1 - \tau})^3) = \frac{1}{3}f(\tau), \ \tau \in [0, 1].
\]

Thus \(BB^* = \frac{1}{3}I, B^*B = \frac{1}{3}I\) and \(\frac{1}{\sqrt{3}}B : L_2([0, 1]) \rightarrow L_2([0, 1])\) is a unitary operator.
4. Find the numerical range of the operator $R : l^2 \to l^2$ defined by

$$Rx = (0, x_1, x_2, \ldots), \ x = (x_1, x_2, \ldots) \in l^2.$$ 

**Solution**

Take an arbitrary $x = (x_1, x_2, \ldots) \in l^2$ such that $\|x\| = 1$. Then

$$|(Rx, x)| = \left| \sum_{k=1}^{\infty} x_k x_{k+1} \right| \leq \sum_{k=1}^{\infty} |x_k| |x_{k+1}| =$$

$$\frac{1}{2} \sum_{k=1}^{\infty} (|x_k|^2 + |x_{k+1}|^2) - \frac{1}{2} \sum_{k=1}^{\infty} (|x_k| - |x_{k+1}|)^2 =$$

$$1 - \frac{1}{2} |x_1|^2 - \frac{1}{2} \sum_{k=1}^{\infty} (|x_k| - |x_{k+1}|)^2.$$ 

The RHS equals 1 iff $|x_1| = 0$ and $|x_{k+1}| = |x_k|$, $\forall k \in \mathbb{N}$, i.e. iff $x = 0$.

Therefore, if $\|x\| = 1$, we obtain $|(Rx, x)| < 1$, i.e.

$$\text{Num}(R) \subset \{ \lambda \in \mathbb{C} : |\lambda| < 1 \}.$$ 

Take an arbitrary $\lambda \in \mathbb{C}$ such that $|\lambda| < 1$ and consider

$$x := \sqrt{1 - |\lambda|^2} (1, \lambda, \lambda^2, \lambda^3, \ldots) \in l^2,$$

i.e.

$$x = (x_1, x_2, \ldots), \ x_k = \sqrt{1 - |\lambda|^2} \lambda^{k-1}, \ k \in \mathbb{N}.$$ 

It is easy to see that $\|x\| = 1$ and

$$(Rx, x) = \sum_{k=1}^{\infty} x_k x_{k+1} = (1 - |\lambda|^2) \sum_{k=1}^{\infty} \lambda^{k-1} \lambda^k = (1 - |\lambda|^2) \lambda \sum_{k=1}^{\infty} |\lambda|^{2(k-1)} =$$

$$= (1 - |\lambda|^2) \lambda \frac{1}{1 - |\lambda|^2} = \lambda.$$

Hence $\lambda \in \text{Num}(R)$ and

$$\text{Num}(R) = \{ \lambda \in \mathbb{C} : |\lambda| < 1 \}.$$
5. Let \( P \) be a non-trivial orthogonal projection \((P \neq 0, I)\). Find its numerical range.

Solution

Since \( \text{Ran}(P) = \text{Ker}(P)^\perp = \text{Ran}(I - P)^\perp \), we have for any \( x \) such that \( \|x\| = 1 \),

\[
\|Px\|^2 \leq \|Px\|^2 + \|(I - P)x\|^2 = \|x\|^2 = 1,
\]

\[
(Px, x) = (Px, Px + (I - P)x) = \|Px\|^2 \in [0, 1].
\]

Consequently

\[
\text{Num}(P) \subset [0, 1].
\]

Since \( P \) is non-trivial, \( \text{Ran}(P) \neq \{0\} \), \( \text{Ker}(P) \neq \{0\} \) and there exist \( y \in \text{Ran}(P) \), \( z \in \text{Ker}(P) \) such that \( \|y\| = 1 = \|z\| \). Take an arbitrary \( t \in [0, 1] \) and consider

\[
x := \sqrt{t}y + \sqrt{1-t}z.
\]

Since \( y \perp z \), it is easy to see that \( \|x\| = 1 \), \( Px = \sqrt{t}y \) and

\[
(Px, x) = \|Px\|^2 = t.
\]

Hence \( t \in \text{Num}(P) \) and

\[
\text{Num}(P) = [0, 1].
\]