OPERATOR THEORY

Solution VI

1. Prove that the norm $\| \cdot \|_p$ on $l^p$, $p \neq 2$, is not induced by an inner product. 
   (*Hint:* Prove that for $x = (1,1,0,\ldots) \in l^p$ and $y = (1,-1,0,\ldots) \in l^p$ the parallelogram law fails.)

   **Solution**

   It is easy to see that
   
   $$\|x + y\|_p = 2 = \|x - y\|_p, \quad \|x\|_p = 2^{1/p} = \|y\|_p.$$  

   Hence the parallelogram law
   
   $$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

   is equivalent to $8 = 4 \cdot 2^{2/p}$, i.e., to $p = 2$. So, if $p \neq 2$, the parallelogram law fails, i.e., the norm $\| \cdot \|_p$ is not induced by an inner product.

2. Prove that the norm $\| \cdot \|_p$, $p \neq 2$ on $C([0,1])$ is not induced by an inner product. (*Hint:* Prove that for functions $f(t) = 1/2 - t$ and

   $$g(t) = \begin{cases} 
   1/2 - t & \text{if } 0 \leq t \leq 1/2, \\
   t - 1/2 & \text{if } 1/2 < t \leq 1,
   \end{cases}$$

   the parallelogram law fails).

   **Solution**

   We have

   $$f(t) + g(t) = \begin{cases} 
   1 - 2t & \text{if } 0 \leq t \leq 1/2, \\
   0 & \text{if } 1/2 < t \leq 1,
   \end{cases}$$

   $$f(t) - g(t) = \begin{cases} 
   0 & \text{if } 0 \leq t \leq 1/2, \\
   1 - 2t & \text{if } 1/2 < t \leq 1,
   \end{cases}$$
\[ \| f + g \|_p = \left( \int_0^{1/2} (1 - 2t)^p dt \right)^{1/p} = \frac{1}{(2(p + 1))^{1/p}}, \]
\[ \| f - g \|_p = \left( \int_{1/2}^1 |1 - 2t|^p dt \right)^{1/p} = \left( \int_{1/2}^1 (2t - 1)^p dt \right)^{1/p} = \frac{1}{(2(p + 1))^{1/p}}, \]
\[ \| f \|_p = \left( \int_0^1 |1/2 - t|^p dt \right)^{1/p} = \left( \int_0^{1/2} (1/2 - t)^p dt + \int_{1/2}^1 (t - 1/2)^p dt \right)^{1/p} = \frac{1}{2(p + 1)^{1/p}} = \| g \|_p. \]

Hence the parallelogram law is equivalent to \( 2 \cdot 2^{-2/p} = 1 \), i.e. to \( p = 2 \). So, if \( p \neq 2 \), the parallelogram law fails, i.e. the norm \( \| \cdot \|_p \) is not induced by an inner product.

3. Let \( \{ e_n \}_{n \in \mathbb{N}} \) be an orthonormal set in an inner product space \( \mathcal{H} \). Prove that
\[
\sum_{n=1}^{\infty} |(x, e_n)(y, e_n)| \leq \| x \| \| y \|, \quad \forall x, y \in \mathcal{H}.
\]

**Solution**

Using the Cauchy–Schwarz inequality for \( l^2 \) and Bessel’s inequality for \( \mathcal{H} \) we obtain
\[
\sum_{n=1}^{\infty} |(x, e_n)(y, e_n)| \leq \left( \sum_{n=1}^{\infty} |(x, e_n)|^2 \right)^{1/2} \left( \sum_{n=1}^{\infty} |(y, e_n)|^2 \right)^{1/2} \leq \| x \| \| y \|. \]
4. Prove that in a complex inner product space $\mathcal{H}$ the following equalities hold:

\[
(x, y) = \frac{1}{N} \sum_{k=1}^{N} \|x + e^{2\pi i k/N} y\| e^{2\pi i k/N} \quad \text{for } N \geq 3,
\]

\[
(x, y) = \frac{1}{2\pi} \int_0^{2\pi} \|x + e^{i\theta} y\|^2 e^{i\theta} d\theta, \quad \forall x, y \in \mathcal{H}.
\]

**Solution**

\[
\frac{1}{N} \sum_{k=1}^{N} \|x + e^{2\pi i k/N} y\|^2 e^{2\pi i k/N} = \frac{1}{N} \sum_{k=1}^{N} (x + e^{2\pi i k/N} y, x + e^{2\pi i k/N} y) e^{2\pi i k/N} =
\]

\[
\frac{1}{N} \sum_{k=1}^{N} \|x\|^2 e^{2\pi i (N+1)/N} - e^{2\pi i/N} e^{2\pi i/N} - 1 + \frac{1}{N} (y,x) \frac{e^{2\pi i (N+1)/N} - e^{2\pi i/N}}{e^{2\pi i/N} - 1} + (x,y) +
\]

\[
\frac{1}{N} \|y\|^2 \left\{ e^{2\pi i (N+1)/N} - e^{2\pi i/N} \right\} e^{2\pi i/N} - 1 = (x,y),
\]

since $e^{2\pi i/N} \neq 1$ and $e^{2\pi i/N} \neq 1$ for $N \geq 3$.

Similarly

\[
\frac{1}{2\pi} \int_0^{2\pi} \|x + e^{i\theta} y\|^2 e^{i\theta} d\theta = \frac{1}{2\pi} \int_0^{2\pi} (x + e^{i\theta} y, x + e^{i\theta} y) e^{i\theta} d\theta =
\]

\[
\frac{1}{2\pi} \|x\|^2 \int_0^{2\pi} e^{i\theta} d\theta + \frac{1}{2\pi} (y, x) \int_0^{2\pi} e^{2i\theta} d\theta + \frac{1}{2\pi} (x, y) \int_0^{2\pi} 1 d\theta +
\]

\[
\frac{1}{2\pi} \|y\|^2 \int_0^{2\pi} e^{i\theta} d\theta = (x,y).
\]

5. Show that $A^\perp = \text{span}A$ for any subset of a Hilbert space.

**Solution**


Claim I: For any subset $M$ of a Hilbert space $H$ the orthogonal complement $M^\perp$ is a closed linear subspace of $H$ (see Proposition 3.15(i)). Indeed, if $z_1, z_2 \in M^\perp$, then

$$(\alpha z_1 + \beta z_2, y) = \alpha (z_1, y) + \beta (z_2, y) = \alpha 0 + \beta 0 = 0, \ \forall y \in M, \ \forall \alpha, \beta \in \mathbb{F}$$

and hence, $\alpha z_1 + \beta z_2 \in M^\perp$. So, $M^\perp$ is a linear subspace of $H$.

Suppose $z \in \text{Cl}(M^\perp)$. Then there exist $z_n \in M^\perp, \ n \in \mathbb{N}$ such that $z_n \to z$ as $n \to +\infty$. Therefore

$$(z, y) = \lim_{n\to +\infty} (z_n, y) = \lim_{n\to +\infty} 0 = 0, \ \forall y \in M,$$

i.e. $z \in M^\perp$. Thus $M^\perp$ is a closed linear subspace of $H$.

Claim II: $A \subset A^{\perp\perp}$ (see Proposition 3.15(ii)). Indeed, for any $x \in A$ we have

$$(x, y) = (y, x) = 0, \ \forall y \in A^\perp,$$

i.e. $x \in A^{\perp\perp}$.

Since $A^{\perp\perp} = (A^\perp)^\perp$ is a closed linear subspace of $H$, we obtain

$$\text{span}A \subset A^{\perp\perp}.$$ 

Now, take any $x \in A^{\perp\perp}$. Since $H = \text{span}A \oplus (\text{span}A)^\perp$, we have

$$x = z + y, \ z \in \text{span}A, \ y \in (\text{span}A)^\perp,$$

and therefore, $(x, y) = (z, y) + \|y\|^2$. Since $y \in A^\perp$, we obtain $0 = \|y\|^2$, i.e. $y = 0$, i.e. $x = z$, i.e. $x \in \text{span}A$. Consequently $A^{\perp\perp} \subset \text{span}A$. Finally,

$$A^{\perp\perp} = \text{span}A.$$

6. Let $M$ and $N$ be closed subspaces of a Hilbert space. Show that $(M + N)^\perp = M^\perp \cap N^\perp$, $(M \cap N)^\perp = \text{Cl}(M^\perp + N^\perp)$.

**Solution**

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Let \( z \in (M + N)^\perp \), i.e.

\[
(z, x + y) = 0, \quad \forall x \in M, \forall y \in N. \tag{1}
\]

Taking \( x = 0 \) or \( y = 0 \) we obtain

\[
(z, x) = 0, \quad \forall x \in M, \quad (z, y) = 0, \quad \forall y \in N, \tag{2}
\]

i.e. \( z \in M^\perp \cap N^\perp \). Hence \((M + N)^\perp \subset M^\perp \cap N^\perp \).

Suppose now \( z \in M^\perp \cap N^\perp \), i.e. (2) holds. Then obviously, (1) holds, i.e \( z \in (M + N)^\perp \). Therefore \( M^\perp \cap N^\perp \subset (M + N)^\perp \). Thus

\[
(M + N)^\perp = M^\perp \cap N^\perp. \tag{3}
\]

Writing (3) for \( M^\perp \) and \( N^\perp \) instead of \( M \) and \( N \) and using Exercise 5, we obtain

\[
(M^\perp + N^\perp)^\perp = M^\perp \cap N^\perp \cap M \cap N,
\]

since \( M \) and \( N \) are closed linear subspaces. Taking the orthogonal complements of the LHS and the RHS and using Exercise 5 again, we arrive at

\[
\text{Cl}(M^\perp + N^\perp) = (M \cap N)^\perp,
\]

since \( M^\perp + N^\perp \) is a linear subspace.

7. Show that \( M := \{ x = (x_n) \in l^2 : \ x_{2n} = 0, \ \forall n \in \mathbb{N} \} \) is a closed subspace of \( l^2 \). Find \( M^\perp \).

**Solution**

Take any \( x, y \in M \). It is clear that for any \( \alpha, \beta \in \mathbb{F} \)

\[
(\alpha x + \beta y)_{2n} = \alpha x_{2n} + \beta y_{2n} = 0,
\]

i.e. \( \alpha x + \beta y \in M \). Hence \( M \) is a linear subspace of \( l^2 \). Let us prove that it is closed.

Take any \( x \in \text{Cl}(M) \). There exist \( x^{(k)} \in M \) such that \( x^{(k)} \to x \) as \( k \to +\infty \).

Since \( x_{2n}^{(k)} = 0 \), we obtain

\[
x_{2n} = \lim_{k \to +\infty} x_{2n}^{(k)} = 0,
\]
i.e. $x \in M$. Hence $M$ is closed.
Further,

$$z \in M^\perp \iff (z, x) = 0, \forall x \in M \iff \sum_{n=0}^\infty z_{2n+1}x_{2n+1} = 0 \text{ for all } x_{2n+1} \in \mathbb{F}, \ n \in \mathbb{N},$$

such that

$$\sum_{n=0}^\infty |x_{2n+1}|^2 < +\infty \iff z_{2n+1} = 0, \forall n = 0, 1, \ldots$$

Therefore

$$M^\perp = \{z = (z_n) \in l^2: z_{2n+1} = 0, \forall n = 0, 1, \ldots \}.$$ 

8. Show that vectors $x_1, \ldots, x_N$ in an inner product space $\mathcal{H}$ are linearly independent iff their Gram matrix $(a_{jk})_{j,k=1}^N = ((x_k, x_j))_{j,k=1}^N$ is nonsingular, i.e. iff the corresponding Gram determinant $\det((x_k, x_j))$ does not equal zero.

Take an arbitrary $x \in \mathcal{H}$ and set $b_j = (x, x_j)$. Show that, whether or not $x_j$ are linearly independent, the system of equations

$$\sum_{k=1}^N a_{jk}c_k = b_j, \ j = 1, \ldots, N,$$

is solvable and that for any solution $(c_1, \ldots, c_N)$ the vector $\sum_{j=1}^N c_jx_j$ is the nearest to $x$ point of $\text{lin}\{x_1, \ldots, x_N\}$.

Solution

Let $c_1, \ldots, c_N \in \mathbb{F}$. It is clear that

$$\sum_{k=1}^N c_kx_k = 0 \iff \left(\sum_{k=1}^N c_kx_k, y\right) = 0, \forall y \in \text{lin}\{x_1, \ldots, x_N\} \iff \left(\sum_{k=1}^N c_kx_k, x_j\right) = 0, \forall j = 1, \ldots, N \iff \sum_{k=1}^N a_{jk}c_k = 0, \forall j = 1, \ldots, N.$$
Therefore

the vectors $x_1, \ldots, x_N$ are linearly independent $\iff$

$$\sum_{k=1}^{N} c_k x_k = 0 \iff c_1 = \cdots = c_N = 0 \iff$$

the system $\sum_{k=1}^{N} a_{jk} c_k = 0$, $j = 1, \ldots, N$ has only

the trivial solution $c_1 = \cdots = c_N = 0 \iff$

$\det((x_k, x_j)) = \det(a_{jk}) \neq 0$.

For any $x \in \mathcal{H}$ and $b_j = (x, x_j), j = 1, \ldots, N$ we have

the vector $\sum_{k=1}^{N} c_k x_k$ is the nearest to $x$ point of

$\text{lin}\{x_1, \ldots, x_N\} \iff$

$x - \sum_{k=1}^{N} c_k x_k \in \text{lin}\{x_1, \ldots, x_N\}^\perp \iff$

$$\left( x - \sum_{k=1}^{N} c_k x_k, x_j \right) = 0, \ j = 1, \ldots, N \iff$$

$$\sum_{k=1}^{N} a_{jk} c_k = b_j, \ j = 1, \ldots, N.$$