# OPERATOR THEORY

## Solution V

1. Let $B \in \mathcal{B}(X)$ and let $T \in \mathcal{B}(Y, X)$ be invertible: $T^{-1} \in \mathcal{B}(X, Y)$. Prove that

$$\sigma(B) = \sigma(T^{-1}BT).$$

Solution

Since $T$ is invertible,

$$\begin{aligned} \lambda \notin \sigma(B) \iff B - \lambda I \text{ is invertible } &\iff T^{-1}(B - \lambda I)T \text{ is invertible} \\ \iff T^{-1}BT - \lambda I \text{ is invertible } &\iff \lambda \notin \sigma(T^{-1}BT), \end{aligned}$$

i.e. $\sigma(B) = \sigma(T^{-1}BT)$.

2. Consider the right-shift operator $R : l^\infty \to l^\infty$ defined by

$$Rx = (0, x_1, x_2, \dots), \quad x = (x_1, x_2, \dots) \in l^\infty.$$

Find the eigenvalues and the spectrum of this operator. Is this operator compact?

Solution

It is clear that $\mathrm{Ker}(R) = \{0\}$, i.e. 0 is not an eigenvalue of $R$. Suppose $\lambda \neq 0$ is an eigenvalue of $R$. Then $Rx = \lambda x$ for some non-zero $x$. So,

$$\begin{aligned} 0 &= \lambda x_1 \\ x_1 &= \lambda x_2 \\ x_2 &= \lambda x_3 \\ &\dots \end{aligned}$$

Solving the last system we obtain $x = 0$. Contradiction! Thus $R$ does not have eigenvalues.

It is clear that $\|Rx\| = \|x\|$ for any $x \in l^\infty$. Therefore $\|R\| = 1$ and $\sigma(R) \subset \{\lambda \in \mathbb{C} : \ |\lambda| \leq 1\}$. Let us take an arbitrary $\lambda \in \mathbb{C}$ such that $0 < |\lambda| < 1$. Suppose $\lambda \notin \sigma(R)$. Then the equation

$$(R - \lambda I)x = y \tag{1}$$

has a unique solution $x \in l^\infty$ for any $y \in l^\infty$. For $y = (1, 0, 0, \dots)$, (1) takes the form

$$\begin{aligned} -\lambda x_1 &= 1 \\ x_1 - \lambda x_2 &= 0 \\ x_2 - \lambda x_3 &= 0 \\ &\dots \end{aligned}$$

Solving the last system we obtain $x_k = -\lambda^{-k}$. Since $|\lambda| < 1$, the element $x = (x_1, x_2, \dots)$ does not belong to $l^\infty$. The obtained contradiction shows that $\lambda \in \sigma(R)$. Hence, $\{\lambda \in \mathbb{C} : \ 0 < |\lambda| < 1\} \subset \sigma(R)$. Taking into account that $\sigma(R)$ is closed we obtain

$$\sigma(R) = \{\lambda \in \mathbb{C} : \ |\lambda| \leq 1\}.$$

The operator $R$ is not compact. This follows from the fact that $R$ is an isometry or from the fact that its spectrum cannot be the spectrum of a compact operator.

3. Consider the set

$$M = \{x \in l^\infty : \ |x_n| \leq n^{-\alpha}, \ n \in \mathbb{N}\} \subset l^\infty,$$

where $\alpha > 0$ is a fixed number. Prove that $M$ is compact.

Solution

It is clear that $M$ is a closed set. Take an arbitrary $\varepsilon > 0$. There exists $N \in \mathbb{N}$ such that $n^{-\alpha} < \varepsilon$, $\forall n > N$. Let $\widetilde{x}^{(l)} = (x_1^{(l)}, \dots, x_N^{(l)}) \in \mathbb{C}^N$, $l = 1, \dots, L$ be a finite $\varepsilon$–net of the set

$$M_N := \{\widetilde{x} = (x_1, \dots, x_N) \in \mathbb{C}^N : \ |x_n| \leq n^{-\alpha}, \ n = 1, \dots, N\}.$$

Such an $\varepsilon$–net can be easily constructed with the help of $\varepsilon$–nets of the disks $\{x_n \in \mathbb{C} : \ |x_n| \leq n^{-\alpha}\}$. (The existence of such an $\varepsilon$–net also follows from the fact that $M_N$ is relatively compact as a bounded subset of the finite–dimensional space $\mathbb{C}^N$.) Now it is easy to see that

$$x^{(l)} := (x_1^{(l)}, \dots, x_N^{(l)}, 0, 0, \dots), \quad l = 1, \dots, L$$

is an $\varepsilon$–net of $M$. Hence $M$ is a closed relatively compact set, i.e. a compact set.

(An alternative proof: it is clear that $M = T(S_\infty)$, where $T \in \mathcal{B}(l^\infty)$,

$$Tx := (x_1, 2^{-\alpha}x_2, 3^{-\alpha}x_3, \dots, n^{-\alpha}x_n, \dots), \quad x = (x_1, x_2, \dots) \in l^\infty$$

and $S_\infty$ is the unit ball of $l^\infty$:

$$S_\infty = \{x \in l^\infty : \ |x_n| \le 1, \ n \in \mathbb{N}\}.$$

It is easy to see that $\|T - T_N\| \to 0$ as $N \to +\infty$, where $T_N$ is a finite rank operator defined by

$$T_N x := (x_1, 2^{-\alpha}x_2, \dots, N^{-\alpha}x_N, 0, 0 \dots), \quad x = (x_1, x_2, \dots) \in l^\infty.$$

Hence $T$ is a compact operator and $M = T(S_\infty)$ is a closed relatively compact set, i.e. a compact set.)

4. Let $g \in C([0,1])$ be a fixed function. Consider the operator $A \in \mathcal{B}(C([0,1]))$ defined by the formula

$$(Au)(s) := g(s)u(s),$$

i.e. the operator of multiplication by $g$. Is this operator compact?

Solution

It is clear that if $g \equiv 0$ then $A$ is compact. Let us prove that t if $g \not\equiv 0$ then $A$ is not compact. Indeed, since $g \not\equiv 0$, there exists a subinterval $[a,b] \subset [0,1]$ such that $m := \min_{s \in [a,b]} |g(s)| > 0$. Consider the sequence $u_n \in C([0,1])$, $n \in \mathbb{N}$, $u_n(s) := \sin(2^n \frac{s-a}{b-a}\pi)$, $s \in [0,1]$. It is clear that $(u_n)$ is a bounded sequence. On the other hand $(Au_n)$ does not have Cauchy subsequences. Indeed, take arbitrary $k, n \in \mathbb{N}$. Assume for definiteness that $k > n$, i.e. $k \ge n+1$. Let $s_n := a + 2^{-(n+1)}(b-a)$. Then $s_n \in [a,b]$ and

$$\|Au_k - Au_n\| = \max_{s \in [0,1]} |g(s)(u_k(s) - u_n(s))| \ge m \max_{s \in [a,b]} |u_k(s) - u_n(s)| \ge$$
$$m|u_k(s_n) - u_n(s_n)| = m|\sin(2^{k-n-1}\pi) - \sin(\pi/2)| = m|0 - 1| = m > 0.$$

Since $(Au_n)$ does not have Cauchy subsequences, $A$ is not compact.

(An alternative proof: according to the solution of Exercise 2, Sheet II, $\sigma(A) = g([0,1])$. If $g \not\equiv 0$ is a constant, then $0 \notin g([0,1])$ and $\sigma(A)$ cannot be the spectrum of a compact operator. If $g$ is nonconstant, then $g([0,1])$ is a connected subset of $\mathbb{C}$ consisting of more than one point and $\sigma(A)$ cannot be the spectrum of a compact operator.)

5. Let $X$ be an infinite-dimensional Banach space and $B, T \in \mathcal{B}(X)$. Which of the following statements are true?
(i) If $BT$ is compact then either $B$ or $T$ is compact.
(ii) If $T^2 = 0$ then $T$ is compact.
(iii) If $T^n = I$ for some $n \in \mathbb{N}$ then $T$ is not compact.

Solution

(i) is false. This follows from the fact that (ii) is false.

(ii) is false. Indeed, let $X = l^p$, $1 \le p \le +\infty$ and

$$Tx = (0, x_1, 0, x_3, 0, x_5, 0, \dots), \quad x = (x_1, x_2, x_3, \dots) \in l^p.$$

Then $T^2 = 0$ and it is easy to see that $T$ is not compact (why?).

(iii) is true. Indeed, suppose $T$ is compact. Then $I = T^n$ is also compact, which is impossible, since $X$ is infinite-dimensional. Contradiction!