OPERATOR THEORY

Solution III

1. Let X be a Banach space and $A, B \in \mathcal{B}(X)$.
(a) Show that if $I - AB$ is invertible, then $I - BA$ is also invertible. [*Hint:* consider $B(I - AB)^{-1}A + I$.]
(b) Prove that if $\lambda \in \sigma(AB)$ and $\lambda \neq 0$, then $\lambda \in \sigma(BA)$.
(c) Give an example of operators A and B such that $0 \in \sigma(AB)$ but $0 \notin \sigma(BA)$.
(d) Show that $\sigma(AB) \bigcup \{0\} = \sigma(BA) \bigcup \{0\}$.
(e) Prove that $r(AB) = r(BA)$.

Solution

(a) Suppose $I - AB$ is invertible. Then since

$$(I - BA)\left(B(I - AB)^{-1}A + I\right) = (I - BA)B(I - AB)^{-1}A + (I - BA) = B(I - AB)(I - AB)^{-1}A + (I - BA) = BA + (I - BA) = I,$$

and

$$\left(B(I - AB)^{-1}A + I\right)(I - BA) = B(I - AB)^{-1}A(I - BA) + (I - BA) = B(I - AB)^{-1}(I - AB)A + (I - BA) = BA + (I - BA) = I,$$

the operator $I - BA$ is also invertible and $(I - BA)^{-1} = B(I - AB)^{-1}A + I$.

(b) Take an arbitrary $\lambda \in \sigma(AB) \setminus \{0\}$. Suppose $BA - \lambda I = -\lambda(I - \frac{1}{\lambda}BA)$ is invertible. Then $I - \frac{1}{\lambda}BA$ is invertible and (a) implies that $I - \frac{1}{\lambda}AB$ is also invertible. Therefore $-\lambda(I - \frac{1}{\lambda}AB) = AB - \lambda I$ is invertible, i.e. $\lambda \notin \sigma(AB)$. This contradiction shows that $BA - \lambda I$ cannot be invertible, i.e. $\lambda \in \sigma(BA)$.

(c) Consider the right and left shift operators $R, L : l^p \to l^p$, $1 \leq p \leq \infty$,

$$Rx = (0, x_1, x_2, \dots), \quad Lx = (x_2, x_3, \dots), \quad \forall x = (x_1, x_2, \dots) \in l^p.$$

It is easy to see that

$$LRx = L(0, x_1, x_2, \dots) = x, \ \ RLx = R(x_2, x_3, \dots) = (0, x_2, x_3, \dots), \ \ \forall x \in l^p.$$

Hence $LR = I$, while $\mathrm{Ran}(RL) \neq l^p$. Therefore LR is invertible while RL is not, i.e. $0 \in \sigma(RL)$ but $0 \notin \sigma(LR)$.

(d) Follows from (b).

(e) Follows from (d) and the definition of the spectral radius.

2. Let X be a Banach space and let operators $A, B \in \mathcal{B}(X)$ commute: $AB = BA$. Prove that $r(A+B) \le r(A) + r(B)$.

<u>Solution</u>

Take an arbitrary $\varepsilon > 0$. The spectral radius formula implies that $\|A^n\| \le (r(A)+\varepsilon)^n$, $\|B^n\| \le (r(B)+\varepsilon)^n$ for sufficiently large $n \in \mathbb{N}$. Therefore there exists $M \ge 1$ such that

$$\|A^n\| \le M(r(A)+\varepsilon)^n, \quad \|B^n\| \le M(r(B)+\varepsilon)^n, \quad \forall n \in \mathbb{N}.$$

Since A and B commute, we have

$$(A+B)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} A^{n-k} B^k.$$

Hence,

$$\|(A+B)^n\| \le \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \left\|A^{n-k}\right\| \left\|B^k\right\| \le$$
$$M^2 \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} (r(A)+\varepsilon)^{n-k} (r(B)+\varepsilon)^k = M^2 (r(A)+r(B)+2\varepsilon)^n$$

for any $n \in \mathbb{N}$. Consequently

$$r(A+B) = \lim_{n\to+\infty} \|(A+B)^n\|^{1/n} \le r(A)+r(B)+2\varepsilon, \quad \forall \varepsilon > 0,$$

i.e. $r(A+B) \le r(A)+r(B)$.

3. Let $k \in C([0,1]\times[0,1])$ be a given function. Consider the operator $B \in \mathcal{B}(C([0,1]))$ defined by the formula

$$(Bu)(s) = \int_0^s k(s,t)u(t)dt.$$

Find the spectral radius of B. What is the spectrum of B? [*Hint:* prove by induction that

$$|(B^n u)(s)| \leq \frac{M^n}{n!} s^n \|u\|_\infty, \quad \forall n \in \mathbb{N},$$

for some constant $M > 0$.]

<u>Solution</u>

Let us prove by induction that

$$|(B^n u)(s)| \leq \frac{M^n}{n!} s^n \|u\|_\infty, \quad \forall s \in [0,1], \quad \forall n \in \mathbb{N}, \tag{1}$$

where

$$M := \max_{(s,t)\in[0,1]^2} |k(s,t)|.$$

For $n = 0$ inequality (1) is trivial. Suppose (1) holds for $n = k$. Then for $n = k + 1$ we have

$$\left|\left(B^{k+1}u\right)(s)\right| = \left|\int_0^s k(s,t)(B^k u)(t)dt\right| \leq \int_0^s |k(s,t)| \left|(B^k u)(t)\right| dt \leq$$
$$M \int_0^s \left|(B^k u)(t)\right| dt \leq M \int_0^s \frac{M^k}{k!} t^k \|u\|_\infty dt = \frac{M^{k+1}}{k!} \|u\|_\infty \int_0^s t^k dt =$$
$$\frac{M^{k+1}}{(k+1)!} s^{k+1} \|u\|_\infty, \quad \forall s \in [0,1].$$

Hence, (1) is proved by induction.
It follows from (1) that

$$\|B^n u\|_\infty \leq \frac{M^n}{n!} \|u\|_\infty, \quad \forall u \in C([0,1]),$$

i.e.

$$\|B^n\| \leq \frac{M^n}{n!}, \quad \forall n \in \mathbb{N}.$$

Therefore

$$r(B) = \lim_{n\to+\infty} \|B^n\|^{1/n} \leq \lim_{n\to+\infty} \frac{M}{(n!)^{1/n}} = 0.$$

Since $r(B) = 0$, $\sigma(B)$ cannot contain nonzero elements. Taking into account that $\sigma(B)$ is nonempty we conclude $\sigma(B) = \{0\}$.