OPERATOR THEORY

Solution II

1. Let $B \in \mathcal{B}(C([0,1]))$ be defined by the formula

$$Bf(t) = tf(t), \quad t \in [0,1].$$

Find $\sigma(B)$ and the set of all eigenvalues of B.

Solution

$\sigma(B) = [0,1]$ and B does not have eigenvalues. This is a special case of Question 2.

2. Let $g \in C([0,1])$ be a fixed function and let $A \in \mathcal{B}(C([0,1]))$ be defined by the formula

$$Af(t) = g(t)f(t), \quad t \in [0,1].$$

Find $\sigma(A)$ and construct effectively the resolvent $R(A;\lambda)$. Find the eigenvalues and eigenvectors of A.

Solution

Let $\lambda \in \mathbb{C}$, $\lambda \notin g([0,1]) := \{g(t) | \ t \in [0,1]\}$. Then since $g \in C([0,1])$, $1/(g-\lambda) \in C([0,1])$ and $A - \lambda I$ has an inverse $R(A;\lambda) = (A-\lambda I)^{-1} \in \mathcal{B}(C([0,1]))$ defined by

$$R(A;\lambda)f(t) = (g(t)-\lambda)^{-1}f(t), \quad t \in [0,1].$$

Hence $\sigma(A) \subset g([0,1])$.
Suppose now $\lambda \in g([0,1])$, i.e. $\lambda = g(t_0)$ for some $t_0 \in [0,1]$. Then $(A - \lambda I)f(t_0) = (g(t_0) - \lambda)f(t_0) = 0$, i.e. $\mathrm{Ran}(A - \lambda I)$ consist of functions vanishing at t_0. Consequently $\mathrm{Ran}(A - \lambda I) \neq C([0,1])$ and $A - \lambda I$ is not invertible. Therefore $g([0,1]) \subset \sigma(A)$. Finally, $\sigma(A) = g([0,1])$.
Take an arbitrary $\lambda \in g([0,1])$. Let $g^{-1}(\lambda) := \{\tau \in [0,1] : \ g(\tau) = \lambda\}$. The equation $Af = \lambda f$, i.e. $(g(t)-\lambda)f(t) = 0$ is equivalent to $f(t) = 0$, $\forall t \in [0,1] \setminus g^{-1}(\lambda)$. If $g^{-1}(\lambda)$ contains an interval of positive length, then it is easy to see that the set $\{f \in C([0,1]) \setminus \{0\} : \ f(t) = 0, \ \forall t \in [0,1] \setminus g^{-1}(\lambda)\}$ is non-empty and coincides with the set of all eigenvectors corresponding to

the **eigenvalue** λ. If $g^{-1}(\lambda)$ does not contain an interval of positive length, then $[0,1] \setminus g^{-1}(\lambda)$ is dense in $[0,1]$ and $f(t) = 0$, $\forall t \in [0,1] \setminus g^{-1}(\lambda)$ implies by continuity that $f \equiv 0$. In this case λ **is not an eigenvalue**.

3. Let $K \subset \mathbb{C}$ be an arbitrary nonempty compact set. Construct an operator $B \in \mathcal{B}(l^p)$, $1 \le p \le \infty$, such that $\sigma(B) = K$.

<u>Solution</u>

Let $\{\lambda_k\}_{k\in\mathbb{N}}$ be a dense subset of K. Consider the operator $B : l^p \to l^p$ defined by

$$Bx = (\lambda_1 x_1, \lambda_2 x_2, \dots, \lambda_k x_k, \dots), \quad \forall x = (x_1, x_2, \dots) \in l^p.$$

Then B is a bounded linear operator and λ_k's are its eigenvalues. (Why?) Consequently $\{\lambda_k\}_{k\in\mathbb{N}} \subset \sigma(B)$. Since $\sigma(B)$ is closed and $\{\lambda_k\}_{k\in\mathbb{N}}$ is dense in K,

$$K \subset \sigma(B).$$

On the other hand, let $\lambda \in \mathbb{C} \setminus K$. Then $d := \inf_{k\in\mathbb{N}} |\lambda_k - \lambda| > 0$ and $B - \lambda I$ has a bounded inverse $(B - \lambda I)^{-1} : l^p \to l^p$ defined by

$$(B - \lambda I)^{-1} x = \left(\frac{1}{\lambda_1 - \lambda} x_1, \frac{1}{\lambda_2 - \lambda} x_2, \dots, \frac{1}{\lambda_k - \lambda} x_k, \dots \right), \quad \forall x = (x_1, x_2, \dots) \in l^p.$$

Hence $\lambda \notin \sigma(B)$. Therefore

$$\sigma(B) \subset K.$$

Finally,

$$\sigma(B) = K$$

4. Let $k \in C([0,1])$ be a given function. Consider the operator $B \in \mathcal{B}(C([0,1]))$ defined by the formula

$$(Bu)(s) = \int_0^s k(t) u(t) dt.$$

Construct effectively (not as a power series!) the resolvent of A. How does this resolvent $R(B;\lambda)$ behave when $\lambda \to 0$?

<u>Solution</u>

It is clear that Bu is continuously differentiable for any $u \in C([0,1])$. Hence, $\mathrm{Ran}(B) \neq C([0,1])$ and B is not invertible, i.e. $0 \in \sigma(B)$.

Suppose now $\lambda \neq 0$. Consider the equation $(B-\lambda I)u = f$, $f \in C([0,1])$, i.e.

$$\int_0^s k(t)u(t)dt - \lambda u(s) = f(s), \quad s \in [0,1]. \tag{1}$$

Suppose this equation has a solution $u \in C([0,1])$. Then

$$\int_0^s k(t)u(t)dt = f(s) + \lambda u(s), \quad s \in [0,1].$$

Therefore $f + \lambda u$ is continuously differentiable and

$$k(s)u(s) = (f(s) + \lambda u(s))', \quad s \in [0,1].$$

If f is continuously differentiable, then

$$k(s)u(s) = f'(s) + \lambda u'(s), \quad s \in [0,1],$$

i.e.

$$u'(s) - \frac{1}{\lambda}k(s)u(s) = -\frac{1}{\lambda}f'(s), \quad s \in [0,1]. \tag{2}$$

Taking $s = 0$ in (1) gives

$$u(0) = -\frac{1}{\lambda}f(0).$$

Solving (2) with this initial condition we obtain

$$u(s) = e^{\frac{1}{\lambda}\int_0^s k(\tau)d\tau}\left(-\frac{1}{\lambda}f(0) - \int_0^s \frac{1}{\lambda}f'(t)e^{-\frac{1}{\lambda}\int_0^t k(\tau)d\tau}dt\right) =$$
$$e^{\frac{1}{\lambda}\int_0^s k(\tau)d\tau}\Bigg(-\frac{1}{\lambda}f(0) - \frac{1}{\lambda}f(t)e^{-\frac{1}{\lambda}\int_0^t k(\tau)d\tau}\Bigg|_0^s -$$
$$\frac{1}{\lambda^2}\int_0^s f(t)k(t)e^{-\frac{1}{\lambda}\int_0^t k(\tau)d\tau}dt\Bigg) = -\frac{1}{\lambda}f(s) - \frac{1}{\lambda^2}\int_0^s f(t)k(t)e^{\frac{1}{\lambda}\int_t^s k(\tau)d\tau}dt.$$

Let

$$A_\lambda f(s) := -\frac{1}{\lambda}f(s) - \frac{1}{\lambda^2}\int_0^s f(t)k(t)e^{\frac{1}{\lambda}\int_t^s k(\tau)d\tau}dt. \tag{3}$$

It is easy to see that $A_\lambda : C([0,1]) \to C([0,1])$ is a bounded linear operator. The above argument shows that if f is continuously differentiable and (1) has a solution $u \in C([0,1])$, then $u = A_\lambda f$. In particular, (1) with $f = 0$ has only a trivial solution $u = 0$, i.e. $\mathrm{Ker}(B - \lambda I) = \{0\}$.
For any $f \in C([0,1])$ and $u = A_\lambda f$ the function

$$f(s) + \lambda u(s) = -\frac{1}{\lambda}\int_0^s f(t)k(t)e^{\frac{1}{\lambda}\int_t^s k(\tau)d\tau}dt$$

is continuously differentiable and

$$(f(s) + \lambda u(s))' = -\frac{1}{\lambda}f(s)k(s) - \frac{k(s)}{\lambda^2}\int_0^s f(t)k(t)e^{\frac{1}{\lambda}\int_t^s k(\tau)d\tau}dt = k(s)A_\lambda f(s) = k(s)u(s)$$

(see (3)). Hence

$$f(s) + \lambda u(s) = \int_0^s k(t)u(t)dt + \mathrm{const.}$$

It follows from (3) that $f(0) + \lambda u(0) = f(0) + \lambda A_\lambda f(0) = 0$. Therefore

$$f(s) + \lambda u(s) = \int_0^s k(t)u(t)dt,$$

i.e. $f = (B - \lambda I)A_\lambda f$, $\forall f \in C([0,1])$, i.e. $(B - \lambda I)A_\lambda = I$. Consequently A_λ is a right inverse of $B - \lambda I$ and $\mathrm{Ran}(B - \lambda I) = C([0,1])$. Since $\mathrm{Ker}(B - \lambda I) = \{0\}$, the operator $B - \lambda I$ is invertible for any $\lambda \neq 0$ and $(B - \lambda I)^{-1} = A_\lambda$. Thus

$$\sigma(B) = \{0\} \quad \text{and} \quad R(B;\lambda) = A_\lambda, \ \forall \lambda \neq 0,$$

where A_λ is given by (3).
It follows from the well known property of the resolvent that $\|R(B;\lambda)\| \geq 1/|\lambda|$, $\forall \lambda \neq 0$. So $\|R(B;\lambda)\| \to \infty$ as $\lambda \to 0$. If $k \equiv 0$, the above inequality becomes an equality. Let us show that if $k \not\equiv 0$, then $\|R(B;\lambda)\|$ grows much faster than $1/|\lambda|$ as $\lambda \to 0$ in a certain direction. Indeed,

$$(R(B;\lambda)1)(s) = (A_\lambda 1)(s) = -\frac{1}{\lambda} - \frac{1}{\lambda^2}\int_0^s k(t)e^{\frac{1}{\lambda}\int_t^s k(\tau)d\tau}dt = -\frac{1}{\lambda} + \frac{1}{\lambda}e^{\frac{1}{\lambda}\int_t^s k(\tau)d\tau}\Big|_0^s = -\frac{1}{\lambda}e^{\frac{1}{\lambda}\int_0^s k(\tau)d\tau}.$$

Further, $k \not\equiv 0$ implies $\int_0^s k(\tau)d\tau \not\equiv 0$. Let

$$C := \max_{[0,1]}\left|\int_0^s k(\tau)d\tau\right| = \left|\int_0^{s_0} k(\tau)d\tau\right| > 0.$$

Then for λ such that $\frac{1}{\lambda}\int_0^{s_0} k(\tau)d\tau > 0$ we have

$$\|R(B;\lambda)\| \geq \|R(B;\lambda)1\| \geq \frac{1}{|\lambda|}e^{C/|\lambda|} \quad \text{as} \quad \lambda \to 0.$$

5. Let $A, B \in \mathcal{B}(X)$. Show that for any $\lambda \in \rho(A)\bigcap\rho(B)$,

$$R(B;\lambda) - R(A;\lambda) = R(B;\lambda)(A-B)R(A;\lambda).$$

<u>Solution</u>

$$R(B;\lambda)(A-B)R(A;\lambda) = R(B;\lambda)((A-\lambda I)-(B-\lambda I))R(A;\lambda) = R(B;\lambda) - R(A;\lambda).$$