OPERATOR THEORY

Solution I

1. Prove that $\mathcal{B}(\mathbb{F}, Y)$ is not a Banach space if Y is not complete. [*Hint:* take a Cauchy sequence (y_n) in Y which does not converge and consider the sequence of operators (B_n),

$$B_n\lambda := \lambda y_n, \ \forall\lambda \in \mathbb{F}.]$$

Solution

Take a Cauchy sequence (y_n) in Y which does not converge and consider the sequence of operators (B_n),

$$B_n\lambda := \lambda y_n, \ \forall\lambda \in \mathbb{F}.$$

It is easy to see that $B_n \in \mathcal{B}(\mathbb{F}, Y)$ and $\|B_n\| = \|y_n\|$, $n \in \mathbb{N}$. Since $(B_n - B_m)\lambda = \lambda(y_n - y_m)$, we have $\|B_n - B_m\| = \|y_n - y_m\|$. Therefore (B_n) is a Cauchy sequence in $\mathcal{B}(\mathbb{F}, Y)$. **Suppose** there exists $B \in \mathcal{B}(\mathbb{F}, Y)$ such that $\|B_n - B\| \to 0$ as $n \to +\infty$. Let $y := B1 \in Y$. Then $\|y_n - y\| = \|B_n 1 - B1\| \le \|B_n - B\| \to 0$ as $n \to +\infty$, i.e. the sequence (y_n) converges to y. This contradiction proves that (B_n) cannot be convergent. Hence $\mathcal{B}(\mathbb{F}, Y)$ is not a Banach space.

2. Give an example of a bounded linear operator A such that $\mathrm{Ran}(A)$ is not closed. [*Hint:* consider the imbedding $X \to Y$, where X is the space $C([0,1])$ equipped with the norm $\|\cdot\|_\infty$ and $Y = L_p([0,1])$ is the completion of the normed space $(C([0,1]), \|\cdot\|_p)$, $1 \le p < \infty$.]

Solution

Let X be the space $C([0,1])$ equipped with the norm $\|\cdot\|_\infty$ and $Y = L_p([0,1])$ be the completion of the normed space $(C([0,1]), \|\cdot\|_p)$, $1 \le p < \infty$. Let $A : X \to Y$ be the imbedding: $Af = f$, $\forall f \in X$. Then $\mathrm{Ran}(A) = C([0,1])$ is dense in $Y = L_p([0,1])$ but does not coincide with it. Therefore $\mathrm{Ran}(A)$ cannot be closed.

3. Give an example of a normed space and an absolutely convergent series in it, which is not convergent.

Solution

Let X be an arbitrary normed non-Banach space (e.g., $X = (C([0,1]), \|\cdot\|_p)$, $1 \le p < \infty$). Then there exists a Cauchy sequence (x_n) in X which **does not converge**. Since (x_n) is Cauchy, for any $k \in \mathbb{N}$ there exists $n_k \in \mathbb{N}$ such that $\|x_n - x_m\| \le 2^{-k}$, $\forall n, m \ge n_k$. Consider the series

$$\sum_{k=1}^{\infty} (x_{n_{k+1}} - x_{n_k})\,. \tag{1}$$

This series is absolutely convergent because

$$\sum_{k=1}^{\infty} \|x_{n_{k+1}} - x_{n_k}\| \le \sum_{k=1}^{\infty} 2^{-k} = 1 < +\infty.$$

On the other hand, (1) is not convergent in X. Indeed, if the sequence of partial sums $\sum_{k=1}^{j} (x_{n_{k+1}} - x_{n_k}) = x_{n_{j+1}} - x_{n_1}$ is convergent as $j \to +\infty$, then so is the sequence $(x_{n_{j+1}})$ and also (x_n), since (x_n) is Cauchy. Contradiction!

4. Let X be the Banach space $C([0,1])$ and Y be the space of all continuously differentiable functions on $[0,1]$ which equal 0 at 0. Both of the spaces are equipped with the norm $\|\cdot\|_\infty$. Show that the linear operator $B : X \to Y$,

$$(Bf)(t) := \int_0^t f(\tau) d\tau,$$

is bounded, one–to–one and onto, but the inverse operator $B^{-1} : Y \to X$ is not bounded. Compare this with the Banach theorem (bounded inverse theorem).

Solution

It is clear that $\|Bf\|_\infty \leq \|f\|_\infty$, $\forall f \in C([0,1])$. So, B is bounded.
Suppose $Bf_1 = Bf_2$, i.e.

$$\int_0^t f_1(\tau)d\tau = \int_0^t f_2(\tau)d\tau, \quad \forall t \in [0,1].$$

Then differentiation gives $f_1(t) = f_2(t)$, $\forall t \in [0,1]$, i.e. $f_1 = f_2$. So, B is one–to–one.
Let φ be continuously differentiable and $\varphi(0) = 0$. Then $\varphi' \in X$ and

$$\varphi(t) = \int_0^t \varphi'(\tau)d\tau, \quad \forall t \in [0,1],$$

i.e. $\varphi = B\varphi'$. So, B is onto.
It follows from the above that B is invertible and

$$B^{-1}\varphi(t) = \varphi'(t), \quad \forall t \in [0,1].$$

It is easy to see that $B^{-1} : Y \to X$ is not bounded. Indeed,

$$\|B^{-1}\sin(nt)\|_\infty = \|n\cos(nt)\|_\infty = n \geq n\|\sin(nt)\|_\infty, \quad \forall n \in \mathbb{N}.$$

This does not contradict the Banach theorem (bounded inverse theorem) because Y equipped with the norm $\|\cdot\|_\infty$ is not a Banach space.

5. Denote by $C^2([0,1])$ the space of twice continuously differentiable functions on the interval $[0,1]$ equipped with the norm

$$\|u\| := \max_{0\leq s\leq 1}|u(s)| + \max_{0\leq s\leq 1}|u'(s)| + \max_{0\leq s\leq 1}|u''(s)|.$$

Let X denote the subspace of $C^2([0,1])$ containing functions satisfying boundary conditions $u(0) = u(1) = 0$. Prove that the operator $A = -\frac{d^2}{ds^2}$ is a bounded operator acting from X to $C([0,1])$.

<u>Solution</u>

The boundedness of $A : X \to C([0,1])$ follows directly from the definition of the norm in X. (Note that $C([0,1])$ is assumed to be equipped with the norm $\|\cdot\|_\infty$.)

6. Show that the operator A from the previous question has a bounded inverse $A^{-1} : C([0,1]) \to X$ and construct it effectively.

Solution

Consider the equation $Au = f$, where $f \in C([0,1])$ is given and $u \in X$ is unknown. It is equivalent to $-\frac{d^2u(s)}{ds^2} = f(s)$, $\forall s \in [0,1]$. Successive integration gives

$$\frac{du(s)}{ds} = -\int_0^s f(\tau)d\tau + c_1,$$
$$u(s) = -\int_0^s \int_0^t f(\tau)d\tau dt + c_1 s + c_0,$$

where c_1 and c_0 are some constants. Now the boundary conditions $u(0) = u(1) = 0$ imply $c_0 = 0$ and

$$c_1 = \int_0^1 \int_0^t f(\tau)d\tau dt.$$

Hence $u = Rf$, where

$$(Rf)(s) := -\int_0^s \int_0^t f(\tau)d\tau dt + s\int_0^1 \int_0^t f(\tau)d\tau dt =$$
$$(s-1)\int_0^s \int_0^t f(\tau)d\tau dt + s\int_s^1 \int_0^t f(\tau)d\tau dt.$$

It is easy to see that $Rf \in X$, $\forall f \in C([0,1])$ and

$$(ARf)(s) = -\frac{d^2}{ds^2}\left(-\int_0^s \int_0^t f(\tau)d\tau dt + s\int_0^1 \int_0^t f(\tau)d\tau dt\right) =$$
$$\frac{d}{ds}\left(\int_0^s f(\tau)d\tau - \int_0^1 \int_0^t f(\tau)d\tau dt\right) = f(s), \quad \forall s \in [0,1],$$
$$(RAu)(s) = -\int_0^s \int_0^t (-u''(\tau))d\tau dt + s\int_0^1 \int_0^t (-u''(\tau))d\tau dt =$$
$$\int_0^s (u'(t) - u'(0))dt - s\int_0^1 (u'(t) - u'(0))dt = u(s) - u(0) - u'(0)s$$
$$-su(1) + su(0) + su'(0) = u(s), \quad \forall s \in [0,1], \quad \forall u \in X,$$

since $u(0) = u(1) = 0$. Hence, $AR = I$ and $RA = I$, i.e. A is invertible and $A^{-1} = R$.
The boundedness of $A^{-1} = R : C([0,1]) \to X$ can be obtained directly from the definition of the norm in X. It also follows immediately from the Banach theorem on the inverse operator. (Note that the completeness of X follows from the well known results on uniform convergence.)