On the classification of factorization structures of low dimension

Javier López Peña



Algebra Department University of Granada (Spain)

ICRA XII Toruń, August 20th 2007

・ロト・西ト・ヨト・ヨー りへの

1/22

Based upon joint works with Gabriel Navarro

- On the classification and properties of noncommutative duplicates, arXiv:math/0612188v1,
- Quantum duplicates of Algebras, (proceedings of the XVIth Integrable Systems and Quantum Symmetries symposium).

and works in progress with Gabriel Navarro and Óscar Cortadellas

On the classification of factorization structures of low dimension

< ロ > < 同 > < 回 > < 回 > < 回 > <









・ロ > < 回 > < 三 > < 三 > 、 三 の へ ()

3 / 22



The solution









ロ > < 日 > < 日 > < 日 > < 日 > < 日 > < 日 > < 日 > < 日 > < 日 > < 1 < つ < 0
</p>

4/22



Factorization structures

Definition (Majid et al.)

We say that X is a *factorization structure* of the algebras A and B if:

• We have $i_A : A \hookrightarrow X$ and $i_B : B \hookrightarrow X$ injective algebra maps.

• The linear map $a \otimes b \mapsto i_A(a) \cdot i_B(B)$ is a linear isomorphism.

Factorization structures are also called **twisted tensor products**.

On the classification of factorization structures of low dimension

イロト イポト イヨト イヨト



Factorization structures

Definition (Majid et al.)

We say that X is a *factorization structure* of the algebras A and B if:

- We have $i_A : A \hookrightarrow X$ and $i_B : B \hookrightarrow X$ injective algebra maps.
- The linear map $a \otimes b \mapsto i_A(a) \cdot i_B(B)$ is a linear isomorphism.

Factorization structures are also called **twisted tensor products**.

On the classification of factorization structures of low dimension

< ロ > < 同 > < 回 > < 回 > .



Factorization structures

Definition (Majid et al.)

We say that X is a *factorization structure* of the algebras A and B if:

- We have $i_A : A \hookrightarrow X$ and $i_B : B \hookrightarrow X$ injective algebra maps.
- The linear map $a \otimes b \mapsto i_A(a) \cdot i_B(B)$ is a linear isomorphism.

Factorization structures are also called *twisted tensor products*.

On the classification of factorization structures of low dimension

< ロ > < 同 > < 回 > < 回 > .



Twisting maps

Definition (Twisting map)

We say that a linear map $R : B \otimes A \longrightarrow A \otimes B$ is a *twisting map* if it satisfies:

$$\textcircled{1} R \circ (B \otimes \mu_{\mathcal{A}}) = (\mu_{\mathcal{A}} \otimes B) \circ (\mathcal{A} \otimes R) \circ (R \otimes \mathcal{A})$$

2 $R \circ (\mu_B \otimes A) = (A \otimes \mu_B) \circ (R \otimes B) \circ (B \otimes R)$

Theorem

The map $\mu_R := (\mu_A \otimes \mu_B) \circ (A \otimes R \otimes B)$ is an associative product in $A \otimes B$ if, and only if, R is a twisting map.

We write $A \otimes_R B$ to denote the algebra $(A \otimes B, \mu_R)$.

・ロト・西ト・ヨト・ヨー もくの

6/22



Twisting maps

Definition (Twisting map)

We say that a linear map $R : B \otimes A \longrightarrow A \otimes B$ is a *twisting map* if it satisfies:

$$\textcircled{0} \hspace{0.1in} R \circ (B \otimes \mu_{\mathcal{A}}) = (\mu_{\mathcal{A}} \otimes B) \circ (\mathcal{A} \otimes R) \circ (R \otimes \mathcal{A})$$

2
$$R \circ (\mu_B \otimes A) = (A \otimes \mu_B) \circ (R \otimes B) \circ (B \otimes R)$$

Theorem

The map $\mu_R := (\mu_A \otimes \mu_B) \circ (A \otimes R \otimes B)$ is an associative product in $A \otimes B$ if, and only if, R is a twisting map.

We write $A \otimes_R B$ to denote the algebra $(A \otimes B, \mu_R)$.

・ロト・西ト・ヨト・ヨー もくの

6/22



Twisting maps

Definition (Twisting map)

We say that a linear map $R : B \otimes A \longrightarrow A \otimes B$ is a *twisting map* if it satisfies:

$$\textcircled{0} \hspace{0.1in} R \circ (B \otimes \mu_{\mathcal{A}}) = (\mu_{\mathcal{A}} \otimes B) \circ (\mathcal{A} \otimes R) \circ (R \otimes \mathcal{A})$$

2
$$R \circ (\mu_B \otimes A) = (A \otimes \mu_B) \circ (R \otimes B) \circ (B \otimes R)$$

Theorem

The map $\mu_R := (\mu_A \otimes \mu_B) \circ (A \otimes R \otimes B)$ is an associative product in $A \otimes B$ if, and only if, R is a twisting map.

We write $A \otimes_R B$ to denote the algebra $(A \otimes B, \mu_R)$.

・ロト・西ト・ヨト・ヨー もくの

6/22

୬ < ୯ 6 / 22

Twisting maps

Definition (Twisting map)

We say that a linear map $R : B \otimes A \longrightarrow A \otimes B$ is a *twisting map* if it satisfies:

$$\textcircled{0} \hspace{0.1in} R \circ (B \otimes \mu_{A}) = (\mu_{A} \otimes B) \circ (A \otimes R) \circ (R \otimes A)$$

2
$$R \circ (\mu_B \otimes A) = (A \otimes \mu_B) \circ (R \otimes B) \circ (B \otimes R)$$

Theorem

The map $\mu_R := (\mu_A \otimes \mu_B) \circ (A \otimes R \otimes B)$ is an associative product in $A \otimes B$ if, and only if, R is a twisting map.

We write $A \otimes_R B$ to denote the algebra $(A \otimes B, \mu_R)$.

On the classification of factorization structures of low dimension

イロト イポト イヨト イヨト



Why twisting maps?

Theorem (Tambara, Majid, Cap-Schichl-Vanžura, ...)

Let (X, i_A, i_B) a factorization structure of A and B, then there is a unique twisting map $R : B \otimes A \to A \otimes B$ such that X is isomorphic to $A \otimes_R B$ as a twisted tensor product.

So, studying factorization structures is equivalent to study twisting maps.

he classification of factorization structures of low dimension

・ロト ・ 同ト ・ ヨト・



Why twisting maps?

Theorem (Tambara, Majid, Cap-Schichl-Vanžura, ...)

Let (X, i_A, i_B) a factorization structure of A and B, then there is a unique twisting map $R : B \otimes A \to A \otimes B$ such that X is isomorphic to $A \otimes_R B$ as a twisted tensor product.

So, studying factorization structures is equivalent to study twisting maps.

On the classification of factorization structures of low dimension

< ロ > < 同 > < 回 > < 回 > .

The solution









・ロ・・白・・ヨ・・ヨ・ シック

8 / 22

Classifying factorization structures

Question

When are ttp's given by different twisting maps isomorphic?

Question

 Given algebras, A and B, may we classify all twisting maps R : B ⊗ A → A ⊗ B?

• And describe the algebras $A \otimes_R B$, up to isomorphism?

・ロト・日本・日本・日本・日本・今日・

9/22

Classifying factorization structures

Question

When are ttp's given by different twisting maps isomorphic?

Question

- Given algebras, A and B, may we classify all twisting maps *R* : B ⊗ A → A ⊗ B?
- And describe the algebras $A \otimes_R B$, up to isomorphism?

On the classification of factorization structures of low dimension

< ロ > < 同 > < 回 > < 回 > < 回 > <

Classifying factorization structures

Question

When are ttp's given by different twisting maps isomorphic?

Question

- Given algebras, A and B, may we classify all twisting maps *R* : B ⊗ A → A ⊗ B?
- And describe the algebras $A \otimes_R B$, up to isomorphism?

On the classification of factorization structures of low dimension

< ロ > < 同 > < 回 > < 回 > < 回 > <

The twisting variety

• A, B (f. dim) algebras

- $\mathcal{T}(A, B) := \{R : B \otimes A \to A \otimes B | R \text{ twisting map}\} \text{ is an affine variety}$
- Isoclasses of the $A \otimes_R B \iff$ "Points" in an orbitspace of $\mathcal{T}(A, B)$.
- Classify these points: very difficult problem in general!
 - No general methods are known, even without taking into account isomorphism classes.
 - Each particular case has to be studied on its own.

・ロト・西ト・西ト・西・ うくぐ

10 / 22

The twisting variety

- A, B (f. dim) algebras
- *T*(*A*, *B*) := {*R* : *B* ⊗ *A* → *A* ⊗ *B*| *R* twisting map} is an affine variety
- Isoclasses of the $A \otimes_R B \iff$ "Points" in an orbitspace of $\mathcal{T}(A, B)$.
- Classify these points: very difficult problem in general!
 - No general methods are known, even without taking into account isomorphism classes.
 - Each particular case has to be studied on its own.

・ロ・・四・・日・・日・・日・

10 / 22

The twisting variety

- A, B (f. dim) algebras
- *T*(*A*, *B*) := {*R* : *B* ⊗ *A* → *A* ⊗ *B*| *R* twisting map} is an affine variety
- Isoclasses of ttp $A \otimes_R B \iff$ "Points" in an orbitspace of $\mathcal{T}(A, B)$.
- Classify these points: very difficult problem in general!
 - No general methods are known, even without taking into account isomorphism classes.
 - Each particular case has to be studied on its own.

・ロト・日・・ヨ・・ヨ・ シック

10 / 22

The twisting variety

- A, B (f. dim) algebras
- *T*(*A*, *B*) := {*R* : *B* ⊗ *A* → *A* ⊗ *B*| *R* twisting map} is an affine variety
- Isoclasses of the $A \otimes_R B \iff$ "Points" in an orbitspace of $\mathcal{T}(A, B)$.
- Classify these points: very difficult problem in general!
 - No general methods are known, even without taking into account isomorphism classes.
 - Each particular case has to be studied on its own.

・日・・四・・川・・日・・日・

10 / 22

10 / 22

The twisting variety

- A, B (f. dim) algebras
- *T*(*A*, *B*) := {*R* : *B* ⊗ *A* → *A* ⊗ *B*| *R* twisting map} is an affine variety
- Isoclasses of ttp $A \otimes_{\mathbb{R}} B \iff$ "Points" in an orbitspace of $\mathcal{T}(A, B)$.
- Classify these points: very difficult problem in general!
 - No general methods are known, even without taking into account isomorphism classes.
 - Each particular case has to be studied on its own.

On the classification of factorization structures of low dimension

・ロット (同) ・ (回) ・ (回)

10 / 22

The twisting variety

- A, B (f. dim) algebras
- *T*(*A*, *B*) := {*R* : *B* ⊗ *A* → *A* ⊗ *B*| *R* twisting map} is an affine variety
- Isoclasses of ttp $A \otimes_{\mathbb{R}} B \iff$ "Points" in an orbitspace of $\mathcal{T}(A, B)$.
- Classify these points: very difficult problem in general!
 - No general methods are known, even without taking into account isomorphism classes.
 - Each particular case has to be studied on its own.

On the classification of factorization structures of low dimension

イロト 不得 トイヨト イヨト 二日

The basics

The problem

The solution



Goal

Classify all existing factorization structures of dimension 4.

For starters, restrict to working over an **algebraically** closed field k

・ロト・白ト・モト・モー しょうくい

11/22

The solution

nan

11/22

Э



Goal

Classify all existing factorization structures of dimension 4.

For starters, restrict to working over an *algebraically* closed field k

the classification of factorization structures of low dimension











12 / 22



The framework

• k an algebraically closed field,

• X a 4-dimensional algebra over k.

Question

Do exist k-algebras A, B, and a twisting map R such that $X \cong A \otimes_R B$?

・ロト・日本・モート ヨー うくぐ

13 / 22



The framework

- k an algebraically closed field,
- X = 4-dimensional algebra over k.

Question

Do exist k-algebras A, B, and a twisting map R such that $X \cong A \otimes_R B$?

・ロト・日本・モート・ヨー うくぐ

13 / 22



The framework

- k an algebraically closed field,
- X a 4-dimensional algebra over k.

Question

Do exist k-algebras A, B, and a twisting map R such that $X \cong A \otimes_R B$?

13 / 22

We take a bottom to top approach:

- If *X* factorizes in a nontrivial way, both *A* and *B* must have dimension 2.
- Over an algebraically closed field, there are only two *k*-algebras of dimension 2:
 - The semisimple algebra k^2 ,
 - The algebra of *dual numbers*, $k[\xi] := k[x]/(x^2)$
- Thus, X must be of one of the three following types:
 - $\begin{array}{c} \bullet & k^2 \otimes_R k^2, \\ \bullet & k^2 \otimes_R k[\xi], \\ \bullet & k[\xi] \otimes_R k[\xi] \end{array}$

・ロト・日下・モート・モー うへぐ

14 / 22

We take a **bottom to top** approach:

- If X factorizes in a nontrivial way, both A and B must have dimension 2.
- Over an algebraically closed field, there are only two *k*-algebras of dimension 2:
 - The semisimple algebra k^2 ,
 - The algebra of **dual numbers**, $k[\xi] := k[x]/(x^2)$.
- Thus, X must be of one of the three following types:
 - $\begin{array}{c} \bullet & k^2 \otimes_R k^2, \\ \bullet & k^2 \otimes_R k[\xi], \\ \bullet & k[\xi] \otimes_R k[\xi] \end{array}$

・ロ・・四・・ヨ・・ヨ・ ヨー うへぐ

14 / 22

We take a bottom to top approach:

- If X factorizes in a nontrivial way, both A and B must have dimension 2.
- Over an algebraically closed field, there are only two *k*-algebras of dimension 2:
 - The semisimple algebra k^2 ,
 - The algebra of *dual numbers*, $k[\xi] := k[x]/(x^2)$.
- Thus, X must be of one of the three following types:
 - $\begin{array}{c} \bullet & k^2 \otimes_R k^2, \\ \bullet & k^2 \otimes_R k[\xi], \\ \bullet & k[\xi] \otimes_R k[\xi] \end{array}$

・ロ・・四・・ヨ・・ヨ・ ヨー うへぐ

14 / 22



We take a bottom to top approach:

- If X factorizes in a nontrivial way, both A and B must have dimension 2.
- Over an algebraically closed field, there are only two *k*-algebras of dimension 2:
 - The semisimple algebra k^2 ,
 - The algebra of *dual numbers*, $k[\xi] := k[x]/(x^2)$.
- Thus, X must be of one of the three following types:
 - $\begin{array}{c} \mathbf{0} \quad k^2 \otimes_{\mathbb{R}} k^2, \\ \mathbf{0} \quad k^2 \otimes_{\mathbb{R}} k[\underline{\epsilon}], \\ \mathbf{0} \quad k[\underline{\epsilon}] \otimes_{\mathbb{R}} k[\underline{\epsilon}] \end{array}$

・ロト・白 ト・ヨ ・ シック

14 / 22

The approach

We take a bottom to top approach:

- If X factorizes in a nontrivial way, both A and B must have dimension 2.
- Over an algebraically closed field, there are only two *k*-algebras of dimension 2:
 - The semisimple algebra k^2 ,
 - The algebra of *dual numbers*, $k[\xi] := k[x]/(x^2)$.
- Thus, X must be of one of the three following types:



On the classification of factorization structures of low dimension

・ロット (同) ・ (回) ・ (回)



The approach

We take a bottom to top approach:

- If X factorizes in a nontrivial way, both A and B must have dimension 2.
- Over an algebraically closed field, there are only two *k*-algebras of dimension 2:
 - The semisimple algebra k^2 ,
 - The algebra of *dual numbers*, $k[\xi] := k[x]/(x^2)$.
- Thus, X must be of one of the three following types:



On the classification of factorization structures of low dimension

The approach

We take a bottom to top approach:

- If X factorizes in a nontrivial way, both A and B must have dimension 2.
- Over an algebraically closed field, there are only two *k*-algebras of dimension 2:
 - The semisimple algebra k^2 ,
 - The algebra of *dual numbers*, $k[\xi] := k[x]/(x^2)$.
- Thus, X must be of one of the three following types:



On the classification of factorization structures of low dimension

・ロト ・ 同ト ・ ヨト・
The approach

We take a bottom to top approach:

- If X factorizes in a nontrivial way, both A and B must have dimension 2.
- Over an algebraically closed field, there are only two *k*-algebras of dimension 2:
 - The semisimple algebra k^2 ,
 - The algebra of *dual numbers*, $k[\xi] := k[x]/(x^2)$.
- Thus, X must be of one of the three following types:



On the classification of factorization structures of low dimension

・ロト ・ 同ト ・ ヨト・

• Ttps $k^n \otimes_R k^2$ are called **noncommutative duplicates**.

Classified by Cibils (2006) with combinatorial techniques.

Theorem (Cibils (2006) + López–Navarro (2007))

Any ttp $k^2 \otimes_R k^2$ is isomorphic to one of the following:

- k⁴,
- The quotient kQ/($Q_{\geq 2}$), where $Q \equiv \circ$
- The algebra of matrices M₂(k),
- The path algebra of the quiver

・ロト・日本・モート・ ヨー うくぐ

15 / 22

- Ttps $k^n \otimes_R k^2$ are called **noncommutative duplicates**.
- Classified by Cibils (2006) with combinatorial techniques.

Theorem (Cibils (2006) + López–Navarro (2007))

Any ttp $k^2 \otimes_R k^2$ is isomorphic to one of the following:

- The quotient kQ/($Q_{>2}$), where $Q \equiv \circ$
- The algebra of matrices M₂(k),
- The path algebra of the quiver

・ロト・日本・モート ヨー うへぐ

15 / 22

- Ttps $k^n \otimes_R k^2$ are called **noncommutative duplicates**.
- Classified by Cibils (2006) with combinatorial techniques.



- Ttps $k^n \otimes_R k^2$ are called **noncommutative duplicates**.
- Classified by Cibils (2006) with combinatorial techniques.



<ロ><日><日><日><日><日><日><日><日><日><日><日><日><日<</p>

$k^2 \otimes_R k^2$. Noncommutative duplicates of k^2

- Ttps $k^n \otimes_R k^2$ are called **noncommutative duplicates**.
- Classified by Cibils (2006) with combinatorial techniques.



- Ttps $k^n \otimes_R k^2$ are called **noncommutative duplicates**.
- Classified by Cibils (2006) with combinatorial techniques.

Theorem (Cibils (2006) + López–Navarro (2007))

Any ttp $k^2 \otimes_R k^2$ is isomorphic to one of the following:

- k⁴,
- The quotient kQ/($Q_{\geq 2}$), where $Q \equiv \circ \circ \circ \circ$
- The algebra of matrices $\mathcal{M}_2(k)$,

• The path algebra of the quiver

◆ロ ▶ ◆母 ▶ ◆臣 ▶ ◆臣 ▶ ○臣 ○ のへぐ

- Ttps $k^n \otimes_R k^2$ are called **noncommutative duplicates**.
- Classified by Cibils (2006) with combinatorial techniques.

Theorem (Cibils (2006) + López–Navarro (2007))

Any ttp $k^2 \otimes_R k^2$ is isomorphic to one of the following:

- k⁴,
- The algebra of matrices $\mathcal{M}_2(k)$,
- The path algebra of the quiver

・ロ・・ 自・・ ヨ・・ ヨ・ うくぐ

15 / 22

On the classification of factorization structures of low dimension

0



$k^2 \otimes_R k[\xi]$. Quantum duplicates of k^2

Ttps kⁿ ⊗_R k[ξ] can be classified by an extension of Cibils' techniques.

Theorem (López–Navarro (2007))

Any ttp $k^2 \otimes_R k[\xi]$ is isomorphic to one of the following:

- The commutative ring (k[ξ])²
- The quotient kQ/($Q_{\geq 2}$), where $Q \equiv \circ \longleftarrow \circ$
- The quotient kQ/(Q>2), where Q \equiv \circ
- The algebra of matrices M₂(k),

・ロット (四)・ (田)・ (日)・

16 / 22



$k^2 \otimes_{\mathcal{R}} k[\xi]$. Quantum duplicates of k^2

Ttps kⁿ ⊗_R k[ξ] can be classified by an extension of Cibils' techniques.

Theorem (López-Navarro (2007))

Any ttp $k^2 \otimes_{\mathbb{R}} k[\xi]$ is isomorphic to one of the following:

- The commutative ring $(k[\xi])^2$,
- The quotient kQ/(Q_{>2}), where Q $\equiv \circ$
- The quotient kQ/($Q_{\geq 2}$), where $Q \equiv \circ$
- The algebra of matrices M₂(k),

< ロ > < 回 > < 三 > < 三 > 、 三 の < で</p>

16 / 22



$k^2 \otimes_{\mathcal{R}} k[\xi]$. Quantum duplicates of k^2

Ttps kⁿ ⊗_R k[ξ] can be classified by an extension of Cibils' techniques.

Theorem (López-Navarro (2007))

Any ttp $k^2 \otimes_{\mathbb{R}} k[\xi]$ is isomorphic to one of the following:

- The commutative ring $(k[\xi])^2$,
- The quotient $kQ/(Q_{>2})$, where $Q \equiv \circ$
- The quotient kQ/($Q_{\geq 2}$), where $Q \equiv \circ$
- The algebra of matrices $\mathcal{M}_2(k)$,

▲□▶▲□▶▲□▶▲□▶ □ のへの

16 / 22



うへで 16/22

$k^2 \otimes_{\mathcal{R}} k[\xi]$. Quantum duplicates of k^2

Ttps kⁿ ⊗_R k[ξ] can be classified by an extension of Cibils' techniques.

Theorem (López-Navarro (2007))

Any ttp $k^2 \otimes_{\mathbb{R}} k[\xi]$ is isomorphic to one of the following:

- The commutative ring $(k[\xi])^2$,
- The quotient $kQ/(Q_{>2})$, where $Q \equiv \circ \checkmark \circ$
- The quotient kQ/($Q_{\geq 2}$), where $Q \equiv \circ$
- The algebra of matrices M₂(k),

On the classification of factorization structures of low dimension

イロト イポト イヨト イヨト



$k^2 \otimes_{\mathcal{R}} k[\xi]$. Quantum duplicates of k^2

Ttps kⁿ ⊗_R k[ξ] can be classified by an extension of Cibils' techniques.

Theorem (López-Navarro (2007))

Any ttp $k^2 \otimes_{\mathbb{R}} k[\xi]$ is isomorphic to one of the following:

- The commutative ring $(k[\xi])^2$,
- The quotient $kQ/(Q_{>2})$, where $Q \equiv \circ \checkmark \circ$
- The quotient $kQ/(Q_{\geq 2})$, where $Q \equiv \circ$

• The algebra of matrices $\mathcal{M}_2(k)$,

On the classification of factorization structures of low dimension

イロト イポト イヨト イヨト



$k^2 \otimes_{\mathcal{R}} k[\xi]$. Quantum duplicates of k^2

Ttps kⁿ ⊗_R k[ξ] can be classified by an extension of Cibils' techniques.

Theorem (López–Navarro (2007))

Any ttp $k^2 \otimes_{\mathbb{R}} k[\xi]$ is isomorphic to one of the following:

- The commutative ring $(k[\xi])^2$,
- The quotient $kQ/(Q_{>2})$, where $Q \equiv \circ \checkmark \circ$
- The quotient $kQ/(Q_{\geq 2})$, where $Q \equiv \circ$
- The algebra of matrices $\mathcal{M}_2(k)$,

On the classification of factorization structures of low dimension

• In this case combinatorial techniques do not work.

- Some brute-force computations are required.
- The variety $\mathcal{T}(k[\xi], k[\xi])$ has two irreducible components.

Theorem (López–Navarro–Cortadellas (2007)) Any ttp $k[\xi] \otimes_R k[\xi]$ is isomorphic to one of the following:

- The commutative ring $k[\xi] \otimes k[\xi] \cong \frac{k[x,y]}{(x^2,y^2)}$
- An algebra of the 1-parameter family X_q , where $X_q := \langle x, y | x^2 = y^2 = 0, yx = qxy \rangle$, for $q \in [-1, 1]$.
- The algebra of matrices M₂(k),

もしゃ 山下 エル・山下 エート

17 / 22

- In this case combinatorial techniques do not work.
- Some brute-force computations are required.
- The variety $\mathcal{T}(k[\xi], k[\xi])$ has two irreducible components.

Theorem (López–Navarro–Cortadellas (2007)) Any the $k[\xi] \otimes_R k[\xi]$ is isomorphic to one of the following:

- The commutative ring $k[\xi] \otimes k[\xi] \cong \frac{k[x,y]}{(x^2,y^2)}$
- An algebra of the 1-parameter family X_q , where $X_q := \langle x, y | x^2 = y^2 = 0, yx = qxy \rangle$, for $q \in [-1, 1)$.
- The algebra of matrices M₂(k),

もしゃ 山下 エル・山下 エート

17 / 22

- In this case combinatorial techniques do not work.
- Some brute-force computations are required.
- The variety $\mathcal{T}(k[\xi], k[\xi])$ has two irreducible components.

Theorem (López–Navarro–Cortadellas (2007)) Any ttp $k[\xi] \otimes_R k[\xi]$ is isomorphic to one of the following:

- The commutative ring $k[\xi] \otimes k[\xi] \cong \frac{k[x,y]}{(x^2,y^2)}$
- An algebra of the 1-parameter family X_q , where $X_q := \langle x, y | x^2 = y^2 = 0, yx = qxy \rangle$, for $q \in [-1, 1)$.
- The algebra of matrices M₂(k),

・ロット (四)・ (田)・ (日)・

17 / 22

- In this case combinatorial techniques do not work.
- Some brute-force computations are required.
- The variety $\mathcal{T}(k[\xi], k[\xi])$ has two irreducible components.

Theorem (López-Navarro-Cortadellas (2007))

Any ttp $k[\xi] \otimes_{\mathbb{R}} k[\xi]$ is isomorphic to one of the following:

- The commutative ring $k[\xi] \otimes k[\xi] \cong \frac{k[x,y]}{(x^2,y^2)}$,
- An algebra of the 1-parameter family X_q , where $X_q := \langle x, y | x^2 = y^2 = 0, yx = qxy \rangle$, for $q \in [-1, 1)$.

• The algebra of matrices $\mathcal{M}_2(k)$,

もしゃ 山下 エル・山下 エート

17 / 22

- In this case combinatorial techniques do not work.
- Some brute-force computations are required.
- The variety $\mathcal{T}(k[\xi], k[\xi])$ has two irreducible components.

Theorem (López-Navarro-Cortadellas (2007))

Any ttp $k[\xi] \otimes_R k[\xi]$ is isomorphic to one of the following:

- The commutative ring $k[\xi] \otimes k[\xi] \cong \frac{k[x,y]}{(x^2,y^2)}$,
- An algebra of the 1-parameter family X_q , where $X_q := \langle x, y | x^2 = y^2 = 0, yx = qxy \rangle$, for $q \in [-1, 1)$.
- The algebra of matrices $\mathcal{M}_2(k)$,

もしゃ 山下 エル・山下 エート

17 / 22

$k[\xi] \otimes_{\mathcal{R}} k[\xi]$. Quantum duplicates of $k[\xi]$

- In this case combinatorial techniques do not work.
- Some brute-force computations are required.
- The variety $\mathcal{T}(k[\xi], k[\xi])$ has two irreducible components.

Theorem (López-Navarro-Cortadellas (2007))

Any ttp $k[\xi] \otimes_R k[\xi]$ is isomorphic to one of the following:

- The commutative ring $k[\xi] \otimes k[\xi] \cong \frac{k[x,y]}{(x^2,y^2)}$,
- An algebra of the 1-parameter family X_q , where $X_q := \langle x, y | x^2 = y^2 = 0, yx = qxy \rangle$, for $q \in [-1, 1)$.

The algebra of matrices M₂(k),

On the classification of factorization structures of low dimension

イロト イポト イヨト イヨト

$k[\xi] \otimes_{\mathcal{R}} k[\xi]$. Quantum duplicates of $k[\xi]$

- In this case combinatorial techniques do not work.
- Some brute-force computations are required.
- The variety $\mathcal{T}(k[\xi], k[\xi])$ has two irreducible components.

Theorem (López-Navarro-Cortadellas (2007))

Any ttp $k[\xi] \otimes_R k[\xi]$ is isomorphic to one of the following:

- The commutative ring $k[\xi] \otimes k[\xi] \cong \frac{k[x,y]}{(x^2,y^2)}$,
- An algebra of the 1–parameter family X_q , where $X_q := \langle x, y | x^2 = y^2 = 0, yx = qxy \rangle$, for $q \in [-1, 1)$.

• The algebra of matrices $\mathcal{M}_2(k)$,

On the classification of factorization structures of low dimension

イロト イポト イヨト イヨト



The algebras of dimension 4



The factorization structures of dimension 4





No apparent pattern relates algebras that can be factorized.

• For any 2–dim. algebra A, there is a twisting map R such that $A \otimes_R A \cong \mathcal{M}_2(k)$ simple.

Conjecture (F. Van Oystaeyen, J. Gómez–Torrecillas)

For any algebra A there exist a twisting map R such that $A \otimes_R A$ is simple.

Remark (Cortadellas-López-Navarro)

If it exists, the resulting algebra is not necessarily unique:

 $\mathbb{C}\otimes_{\mathcal{R}}\mathbb{C}\cong\mathcal{M}_2(\mathbb{R}),\qquad\mathbb{C}\otimes_{\mathcal{S}}\mathbb{C}\cong\mathbb{H}$

▲□▶▲□▶▲□▶▲□▶ □□ - つへで

20 / 22



- No apparent pattern relates algebras that can be factorized.
- For any 2–dim. algebra A, there is a twisting map R such that $A \otimes_R A \cong \mathcal{M}_2(k)$ simple.

Conjecture (F. Van Oystaeyen, J. Gómez-Torrecillas)

For any algebra A there exist a twisting map R such that $A \otimes_R A$ is simple.

Remark (Cortadellas-López-Navarro)

If it exists, the resulting algebra is not necessarily unique:

 $\mathbb{C}\otimes_{\mathcal{R}}\mathbb{C}\cong\mathcal{M}_{2}(\mathbb{R}),\qquad\mathbb{C}\otimes_{\mathcal{S}}\mathbb{C}\cong\mathbb{H}$

・ロ・・ 自・・ ヨ・・ ヨ・ うくぐ

20 / 22



- No apparent pattern relates algebras that can be factorized.
- For any 2–dim. algebra A, there is a twisting map R such that $A \otimes_R A \cong \mathcal{M}_2(k)$ simple.

Conjecture (F. Van Oystaeyen, J. Gómez-Torrecillas)

For any algebra A there exist a twisting map R such that $A \otimes_R A$ is simple.

Remark (Cortadellas-López-Navarro)

If it exists, the resulting algebra is not necessarily unique:

 $\mathbb{C}\otimes_{R}\mathbb{C}\cong\mathcal{M}_{2}(\mathbb{R}),\qquad\mathbb{C}\otimes_{S}\mathbb{C}\cong\mathbb{H}$

・ロ・・ 自・・ ヨ・・ ヨ・ うくぐ

20 / 22



- No apparent pattern relates algebras that can be factorized.
- For any 2–dim. algebra A, there is a twisting map R such that $A \otimes_R A \cong \mathcal{M}_2(k)$ simple.

Conjecture (F. Van Oystaeyen, J. Gómez-Torrecillas)

For any algebra A there exist a twisting map R such that $A \otimes_R A$ is simple.

Remark (Cortadellas-López-Navarro)

If it exists, the resulting algebra is not necessarily unique:

 $\mathbb{C}\otimes_{\mathcal{R}}\mathbb{C}\cong\mathcal{M}_2(\mathbb{R}),\qquad\mathbb{C}\otimes_{\mathcal{S}}\mathbb{C}\cong\mathbb{H}$

<ロト < 団 > < 目 > < 目 > < 目 > の < の<</p>

20 / 22

What if k is not algebraically closed?

- Even if *k* is not alg. closed, all the above are valid factorizations.
- However, there may be more valid factors.
- We have to consider all \bar{k} quadratic field extensions of k.
- New cases to consider:
 - $k^2 \otimes_R \bar{k}$ (nc. duplicates of quadratic extensions),
 - $k[\xi] \otimes_{\mathbb{R}} \bar{k}$ (quantum duplicates of quad. extensions),
 - $\bar{k} \otimes_R \bar{k}'$ (products of two quad. extensions).
- Work in progress going along these lines.

21/22

What if k is not algebraically closed?

- Even if k is not alg. closed, all the above are valid factorizations.
- However, there may be *more valid factors*.
- We have to consider all \bar{k} quadratic field extensions of k.
- New cases to consider:
 - $k^2 \otimes_R \bar{k}$ (nc. duplicates of quadratic extensions),
 - $k[\xi] \otimes_R \bar{k}$ (quantum duplicates of quad. extensions),
 - $\bar{k} \otimes_R \bar{k}'$ (products of two quad. extensions).
- Work in progress going along these lines.

(ロ)

21/22

What if k is not algebraically closed?

- Even if k is not alg. closed, all the above are valid factorizations.
- However, there may be *more valid factors*.
- We have to consider all \bar{k} quadratic field extensions of k.
- New cases to consider:

k² ⊗_R k̄ (nc. duplicates of quadratic extensions),
k[ξ] ⊗_R k̄ (quantum duplicates of quad. extensions),
k̄ ⊗_R k̄' (products of two quad. extensions).

• Work in progress going along these lines.

On the classification of factorization structures of low dimension

イロト イポト イヨト イヨト

What if k is not algebraically closed?

- Even if *k* is not alg. closed, all the above are valid factorizations.
- However, there may be *more valid factors*.
- We have to consider all \bar{k} quadratic field extensions of k.
- New cases to consider:
 - $k^2 \otimes_R \overline{k}$ (nc. duplicates of quadratic extensions),
 - $k[\xi] \otimes_R \overline{k}$ (quantum duplicates of quad. extensions),
 - $\bar{k} \otimes_R \bar{k}'$ (products of two quad. extensions).
- Work in progress going along these lines.

On the classification of factorization structures of low dimension

・ロト ・ 同ト ・ ヨト・

What if k is not algebraically closed?

- Even if *k* is not alg. closed, all the above are valid factorizations.
- However, there may be *more valid factors*.
- We have to consider all \bar{k} quadratic field extensions of k.
- New cases to consider:
 - $k^2 \otimes_R \overline{k}$ (nc. duplicates of quadratic extensions),
 - $k[\xi] \otimes_{\mathbb{R}} k$ (quantum duplicates of quad. extensions),
 - $\bar{k} \otimes_R \bar{k}'$ (products of two quad. extensions).
- Work in progress going along these lines.

On the classification of factorization structures of low dimension

・ロト ・ 同ト ・ ヨト・

୬ ଏ ୯ 21 / 22

What if k is not algebraically closed?

- Even if *k* is not alg. closed, all the above are valid factorizations.
- However, there may be *more valid factors*.
- We have to consider all \bar{k} quadratic field extensions of k.
- New cases to consider:
 - $k^2 \otimes_R \overline{k}$ (nc. duplicates of quadratic extensions),
 - $k[\xi] \otimes_R \overline{k}$ (quantum duplicates of quad. extensions),
 - $k \otimes_R k'$ (products of two quad. extensions).
- Work in progress going along these lines.

On the classification of factorization structures of low dimension

What if k is not algebraically closed?

- Even if *k* is not alg. closed, all the above are valid factorizations.
- However, there may be *more valid factors*.
- We have to consider all \bar{k} quadratic field extensions of k.
- New cases to consider:
 - $k^2 \otimes_R \overline{k}$ (nc. duplicates of quadratic extensions),
 - $k[\xi] \otimes_R \overline{k}$ (quantum duplicates of quad. extensions),
 - $\bar{k} \otimes_R \bar{k}'$ (products of two quad. extensions).
- Work in progress going along these lines.

On the classification of factorization structures of low dimension

୬ ଏ ୯ 21 / 22

What if k is not algebraically closed?

- Even if *k* is not alg. closed, all the above are valid factorizations.
- However, there may be *more valid factors*.
- We have to consider all \bar{k} quadratic field extensions of k.
- New cases to consider:
 - $k^2 \otimes_R \overline{k}$ (nc. duplicates of quadratic extensions),
 - $k[\xi] \otimes_{\mathbb{R}} \overline{k}$ (quantum duplicates of quad. extensions),
 - $\bar{k} \otimes_R \bar{k}'$ (products of two quad. extensions).
- Work in progress going along these lines.

On the classification of factorization structures of low dimension





Thanks for your attention!

On the classification of factorization structures of low dimension

イロン イロン イヨン



1