On inner deformations of algebras

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Departamento de Álgebra Universidad de Granada (España)

Noncommutative Rings and Geometry, in honour of Freddy Van Oystaeyen Almería, September 21st 2007

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Based upon joint work with Florin Panaite and Freddy Van Oystaeyen

General twisting of algebras, Adv. Math. 212 (1), 315–337 (2007).



The final answer?











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Inner deformations

Concept of inner deformations.

- Start with an algebra A
- Keep the underlying vector space
- Endow it with a new product (related with the old one)
- 2 There are many examples of inner deformations
 - Twisted tensor products
 - Twisted bialgebras
 - Drinfeld twist for an H-module algebra
 - Deformation via neat elements
 - Deformations by *R-matrices*

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Definition (Twisting map)

A linear map $R: B \otimes A \longrightarrow A \otimes B$ is a *twisting map* if it satisfies:

$$\blacksquare \ R \circ (B \otimes \mu_A) = (\mu_A \otimes B) \circ (A \otimes R) \circ (R \otimes A)$$

2 $R \circ (\mu_B \otimes A) = (A \otimes \mu_B) \circ (R \otimes B) \circ (B \otimes R)$

Theorem

The map $\mu_R := (\mu_A \otimes \mu_B) \circ (A \otimes R \otimes B)$ is an associative product in $A \otimes B$ if, and only if, R is a twisting map.

The algebra $A \otimes_R B := (A \otimes B, \mu_R)$ is called a **twisted tensor product** of A and B.

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L-R twisting datum

• A an H-bimodule algebra with actions

$$\pi_l(h \otimes a) = h \cdot a, \qquad \pi_r(a \otimes h) = a \cdot h,$$

and an H-bicomodule algebra, with coactions

$$\psi_l(\boldsymbol{\textit{a}}) := \boldsymbol{\textit{a}}_{[-1]} \otimes \boldsymbol{\textit{a}}_{[0]}, \qquad \psi_l(\boldsymbol{\textit{a}}) := \boldsymbol{\textit{a}}_{<0>} \otimes \boldsymbol{\textit{a}}_{<1>},$$

satisfying some technical compatibility conditions.

• Define a new multiplication on A by

$$a \bullet a' := (a_{[0]} \cdot a'_{<1>})(a_{[-1]} \cdot a'_{<0>})$$

• Then $(A, \bullet, 1)$ is an associative unital algebra.

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Fedosov product for DG algebras

• $(\Omega = \bigoplus_{n>0} \Omega^n, d)$ differential graded algebra.

• The *Fedosov product* is given by

 $\omega \circ \zeta = \omega \zeta - (-1)^{|\omega|} d\omega \, d\zeta$

• Gives a new (\mathbb{Z}_2 -graded) algebra structure on Ω .

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First approach

The final answer?











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• All these deformations are built in the same way:

- Start with an algebra (A, μ)
- Define some map $T : A \otimes A \to A \otimes A$
- Define a new product by $\mu_T := \mu \circ T$.

Question

Is it possible to obtain the associativity just out of some properties of the map T?

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First approach: R-matrices

Definition (Borcherds)

An *R*-matrix for an algebra A is a map $T : A \otimes A \rightarrow A \otimes A$ such that

$$T(1 \otimes a) = 1 \otimes a, \quad T(a \otimes 1) = a \otimes 1,$$

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Theorem (Borcherds)

If T is an R–matrix for A, then $\mu_A \circ T$ is an associative product.

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The need for something else

- R-matrices provide a set of sufficient conditions for building inner deformations
- But they are not enough
- Twisted tensor product are **NOT** R-matrices.

Question

Is possible to find an approach similar to R-matrices that includes twisted tensor products?

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Theorem

The map $\mu \circ T$ is associative, with the same unit 1.

T is called a *twistor* for *D*.

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- Product achieved via L–R–twisting datum
- Drinfeld cocycle twist of a module algebra
- Deformation of a bialgebra via neat elements
- Deformation of algebras with a differential

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- Braided quantum groups
- The square of a ribbon operator

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Examples of Non-Twistors

- Most R-matrices
- Fedosov product on DG algebras
- Braided quantum groups
- The square of a ribbon operator

Question

Can we find something more general, containing twistors and all the above things?





- The problem
- **First approach**

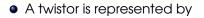


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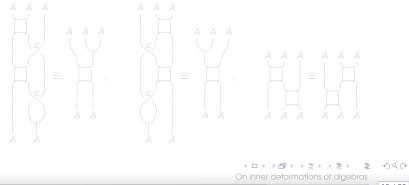
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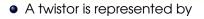
Braiding knotation for twistors



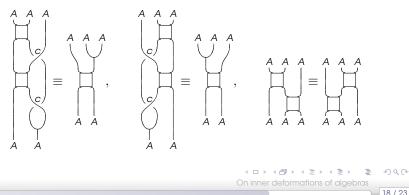
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Braiding knotation for twistors



Twistor conditions are written as



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- Braiding notation gives us a "general shape" for twistor conditions
- Makes easy to spot points where axioms can be weakened
- Allows a categorical formulation
- Leads to the "correct" definition

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Pseudotwistors (I)

$\bullet \ {\cal C}$ a (strict) monoidal category,

- (A, μ, u) an algebra in C,
- $T : A \otimes A \rightarrow A \otimes A$ morphism in C such that $T \circ (u \otimes A) = u \otimes A$ and $T \circ (A \otimes u) = A \otimes u$.
- $\widetilde{T}_1, \widetilde{T}_2 : A \otimes A \otimes A \rightarrow A \otimes A \otimes A$ morphisms in C such that

 $(A \otimes \mu) \circ \widetilde{T}_{1} \circ (T \otimes A) = T \circ (A \otimes \mu),$ $(\mu \otimes A) \circ \widetilde{T}_{2} \circ (A \otimes T) = T \circ (\mu \otimes A),$ $\widetilde{T}_{1} \circ (T \otimes A) \circ (A \otimes T) = \widetilde{T}_{2} \circ (A \otimes T) \circ (T \otimes A).$

• Then $(A, \mu \circ T, u)$ is also an algebra in C.

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The motivation

The problem

First approach

The final answer?

Pseudotwistors (II)

Definition

- The morphism T is called a *pseudotwistor*,
- The morphisms T_1 , T_2 are called the *companions* of *T*.



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Pseudotwistors (II)

Definition

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- The morphisms \tilde{T}_1 , \tilde{T}_2 are called the *companions* of *T*.



Examples of pseudotwistors

- Every twistor 7 is a pseudotwistor, with companions $\widetilde{T}_1 = \widetilde{T}_2 = T_{13}$
- Twistors in a braided category, with companions $\widetilde{T}_1 = \widetilde{T}_2 = T_{13}(c)$
 - This case includes Fedosov products for DGA's
- $G = (A, \mu, \Delta, \varepsilon, S, \sigma)$ a braided quantum group
 - All maps $\sigma_n^{-1} \circ \sigma$ are pseudotwistors, with companions $\tilde{T}_1(\sigma_n), \tilde{T}_2(\sigma_n)$.
 - The multiplications associated to these pseudotwistors are the μn's defined by Durdevich
- *T* a *bijective R-matrix*, then *T* is a pseudotwistor, with companions $\tilde{T}_1 = T_{12} \circ T_{13} \circ T_{12}^{-1}$ and $\tilde{T}_2 = T_{23} \circ T_{13} \circ T_{23}^{-1}$.

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Happy birthday, Fred!