# Calibrated Submanifolds

## Problem Sheet 4

1. For maps $z_1, \ldots, z_n : \mathbb{R} \to \mathbb{C}$ define

$$N = \{(t_1 z_1(s), \ldots, t_n z_n(s)) \in \mathbb{C}^n \,:\, s \in \mathbb{R},\, t_1, \ldots, t_n \in \mathbb{R},\, \sum_{j=1}^{n} t_j^2 = 1\}.$$

   (a) Show that $N$ is special Lagrangian with phase $i$ (so calibrated by $\operatorname{Im}\Omega$) if and only if

$$\overline{z_j}\frac{\mathrm{d}z_j}{\mathrm{d}s} = i f_j \overline{z_1 \ldots z_n}$$

   for positive real functions $f_j$.

   (b) Suppose that $f_j = 1$ for all $j$. Write $z_j = r_j e^{i\theta_j}$, let $\theta = \sum_{j=1}^{n} \theta_j$ and suppose that $z_j(0) = c_j > 0$. Show that $r_j^2 = c_j^2 + u$ for some function $u$ with $u(0) = 0$ and $r_1 \ldots r_n \cos\theta = c_1 \ldots c_n$.

   (c) Suppose that $u = t^2$ and hence show that

$$\theta_j(t) = \int_0^t \frac{a_j \mathrm{d}t}{(1 + a_j t^2)\sqrt{\frac{1}{t^2}\left((1 + a_1 t^2) \ldots (1 + a_n t^2) - 1\right)}}$$

   where $a_j = c_j^{-2}$. Deduce that $\theta \to \pm\frac{\pi}{2}$ as $t \to \pm\infty$.

   (d) Hence describe the asymptotics of $N$ for $n \geq 3$.

2. Fix $t > 0$ and define

$$f : X = \{(a_1, \ldots, a_n) \in \mathbb{R}^n \,:\, a_j \geq 0\} \to Y = \{(\theta_1, \ldots, \theta_n) \in \mathbb{R}^n \,:\, \theta_j \geq 0, \sum_{j=1}^{n} \theta_j < \frac{\pi}{2}\}$$

   by $f(a_1, \ldots, a_n) = (\theta_1, \ldots, \theta_n)$ where

$$\theta_j = \int_0^t \frac{a_j \mathrm{d}t}{(1 + a_j t^2)\sqrt{\frac{1}{t^2}\left((1 + a_1 t^2) \ldots (1 + a_n t^2) - 1\right)}}.$$

   (a) Show that if $n = 1$, $f : X \to Y$ is surjective.

   (b) For $\theta \in (0, \frac{\pi}{2})$ define $H_\theta = \{(\theta_1, \ldots, \theta_n) \in Y \,:\, \sum_{j=1}^{n} \theta_j = \theta\}$. Use Question 1 to show that $f^{-1}(H_\theta) \subseteq S_\theta = \{(a_1, \ldots, a_n) \in X \,:\, (1 + a_1 t^2) \ldots (1 + a_n t^2) = \cos^{-2} t\}$.

   (c) Show by induction on $n$ that $f : S_\theta \to H_\theta$ is surjective. (You may use the fact that if the degree of $f : \partial S_\theta \to \partial H_\theta$ is 1 then the degree of $f : S_\theta \to H_\theta$ is 1.)

   (d) Given any plane $\{(e^{i\theta_1} x_1, \ldots e^{i\theta_n} x_n) \,:\, (x_1, \ldots, x_n) \in \mathbb{R}^n\}$ where $(\theta_1, \ldots, \theta_n) \in Y$, $\theta_j \neq 0$ for all $j$, show that there exists $N$ as in Question 1 which intersects the plane in a hypersurface.

3. Let $N$ be a compact associative submanifold in a $\mathrm{G}_2$ manifold $(M, \varphi)$. Let $\varphi_t$ for $t \in (-\epsilon, \epsilon)$ be a family of torsion-free $\mathrm{G}_2$ structures on $M$ such that $\varphi_0 = \varphi$.

   (a) Show that if $A$ is rigid then there exists $\delta \in (0, \epsilon]$ so that for all $t \in (0, \delta)$ there exists a compact deformation $N_t$ of $N$ so that $N_t$ is associative in $(M, \varphi_t)$.

   (b) Suppose that $\operatorname{Ker} \not{D}$ on $\nu(A)$ is 1-dimensional. Find a condition on $\varphi_t$ which ensures that the same conclusion as (a) holds.

4. Decompose $\operatorname{Im}\mathbb{O} = \operatorname{Im}\mathbb{H} \oplus \mathbb{H}$, let $f : \operatorname{Im}\mathbb{H} \to \mathbb{H}$ be smooth and let $N = \operatorname{Graph}(f)$. Find the condition on $f$ for $N$ to be associative.