Calibrated Submanifolds

Problem Sheet 3

1. Let X, Y be Banach spaces, let $U \subseteq X$ be an open set with $0 \in U$ and let $P : U \to Y$ be a smooth map with $P(0) = 0$. Suppose that $L_0P : X \to Y$ is Fredholm; so, L_0P has finite-dimensional kernel and there exists a finite-dimensional complement (called the cokernel) of the range of L_0P in Y (and the range of L_0P is closed in Y).

   (a) Show that there exist finite-dimensional vector spaces $\mathcal{I}$ and $\mathcal{O}$, an open set $\mathcal{U} \subseteq \mathcal{I}$ containing 0 and a smooth map $\pi : \mathcal{U} \to \mathcal{O}$ with $\pi(0) = 0$ such that $P^{-1}(0)$ is locally homeomorphic to $\pi^{-1}(0)$.

   (b) Deduce that the expected dimension of P is the index of L_0P, which is the difference in the dimensions of the kernel and cokernel.

2. Let X, Y, Z be Banach spaces, with Y finite-dimensional, and let $P : U \subseteq X \times Y \to Z$ with $P(0, 0) = 0$. Suppose further that if we write $L_0P(x, y) = L_X(x) + L_Y(y)$ then $L_X : X \to Z$ is surjective with finite-dimensional kernel K.

   Show that there exist open sets $V \subseteq Y$ and $W \subseteq X$, both containing 0, and a smooth map $G : V \to W$ so that $P(G(y), y) = 0$ for all $y \in V$.

3. Let N be a compact special Lagrangian in a Calabi–Yau n-fold (M, ω, Ω). Let (ω_t, Ω_t) be a family of Calabi–Yau structures for $t \in (-\epsilon, \epsilon)$ with $(\omega_0, \Omega_0) = (\omega, \Omega)$ such that $[\omega_t] = 0$ in $H^2(N)$ and $[\operatorname{Im} \Omega_t] = 0$ in $H^n(N)$.

   By using Question 2, show that there exists $\delta \in (0, \epsilon]$ such that there exists a compact deformation N_t of N for all $t \in (0, \delta)$ with N_t special Lagrangian in (M, ω_t, Ω_t).

4. Let $u_1, u_2, u_3, u_4 \in \mathbb{R}^7 \cong \operatorname{Im} \mathbb{O}$ (the imaginary octonions). We can define a cross product on $\mathbb{R}^7 \cong \operatorname{Im} \mathbb{O}$ via the formula $\varphi(x, y, z) = \langle x \times y, z \rangle$ and then multiplication of $x, y \in \operatorname{Im} \mathbb{O}$ is given by $xy = -\langle x, y \rangle + x \times y$, so $\operatorname{Re}(xy) = -\langle x, y \rangle$ and $\operatorname{Im}(xy) = x \times y$.

   (a) Define $[u_1, u_2, u_3] = (u_1u_2)u_3 - u_1(u_2u_3)$. Show that $\operatorname{Span}\{u_1, u_2, u_3\}$ is calibrated by φ if and only if $[u_1, u_2, u_3] = 0$.

   (b) Show that $\operatorname{Span}\{u_1, u_2, u_3, u_4\}$ is calibrated by $*\varphi$ if and only if $\varphi(u_i, u_j, u_k) = 0$ for all i, j, k.