Calibrated Submanifolds

Problem Sheet 1

1. Let $U \subseteq \mathbb{R}^2$ be an open set with compact closure and let $f : U \to \mathbb{R}$ be such that $N = \mathrm{Graph}(f)$ is a minimal surface in $\mathbb{R}^3$. Let x, y be coordinates on $\mathbb{R}^2$.

   (a) Show that

   $$\mathrm{div}\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right) = 0$$

   and, equivalently, that

   $$(1+f_y^2)f_{xx} + (1+f_x^2)f_{yy} - 2f_xf_yf_{xy} = 0.$$

   Let ν be the upward pointing unit normal on N and define a 2-form η on $U \times \mathbb{R}$ by $\eta(u,v) = \nu.(u \times v)$ for vectors $u, v \in \mathbb{R}^3$.

   (b) Show that η is a calibration.

   (c) Deduce that if N' is any surface in $U \times \mathbb{R}$ with $\partial N' = \partial N$ then $\mathrm{Vol}(N') \geq \mathrm{Vol}(N)$.

2. (a) Let N be a submanifold in $\mathbb{R}^n$. Show that N is minimal if and only if the coordinate functions x_i on $\mathbb{R}^n$ restricted to N are harmonic.

   The maximum principle for harmonic functions states that a harmonic function on a compact set attains its maximum on the boundary.

   (b) By using the maximum principle and (a), or otherwise, show that there are no compact minimal submanifolds in $\mathbb{R}^n$.

3. Let M be Enneper's surface:

   $$M = \left\{\left(s - \frac{s^3}{3} + st^2, -t - s^2t + \frac{t^3}{3}, s^2 - t^2\right) \in \mathbb{R}^3 \,:\, s, t \in \mathbb{R}\right\}.$$

   Show that M is a minimal surface which is not embedded.

4. Let x, y be coordinates on $\mathbb{R}^2$.

   (a) Let $\eta = \frac{1}{\sqrt{x^2+y^2}}(x\mathrm{d}x + y\mathrm{d}y)$ on $\mathbb{R}^2 \setminus \{0\}$. Show that η is a calibration and describe its calibrated submanifolds.

   (b) Let $\eta = \frac{1}{\sqrt{x^2+y^2}}(\mathrm{d}x - \frac{x}{y}\mathrm{d}y)$ for $y > 0$. Show that η is a calibration on the upper-half plane with the hyperbolic metric $\frac{\mathrm{d}x^2+\mathrm{d}y^2}{y^2}$ and describe its calibrated submanifolds.