'What should finite time singularities of LMCF look like?'

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Based on 'Conjectures on Bridgeland stability...', EMS Surveys in Math. Sci. 2 (2015), arXiv:1401.4949.

Apologies: — No theorems, only conjectures / claims
— Little new work since January 2014.

1. Generalities about Lagrangian MCF.

Set-up: (M, J, g, Ω) Calabi-Yau m-fold.
g Ricci flat Kähler, Ω holomorphic $(m,0)$ form,
$|\Omega|_g = 2^{m/2}$. ω Kähler form of g.

$L \subset M$ Lagrangian in (M, ω), embedded or immersed.
Usually L is compact, oriented.
Then $\Omega|_L = e^{i\theta} dV_L$, dV_L volume form of $(L, g|_L)$,
$e^{i\theta}: L \longrightarrow U(1)$ smooth, the phase function of L.
L is special Lagrangian if $e^{i\theta}$ is constant.
L is Maslov zero if $[d\theta] = 0$ in $H^1_{dR}(L, \mathbb{R})$.

L is graded if we have smooth $\theta : L \to \mathbb{R}$ lifting $e^{i\theta} : L \to U(1)$. Graded $\Rightarrow$ Maslov zero.

If L is graded, the phase variation is

$\mathrm{var}\, L = \max_L \theta - \min_L \theta$. Then $\mathrm{var}\, L = 0$ iff L special Lagrangian.

L is almost calibrated if $\mathrm{var}\, L < \pi$.

The mean curvature of L is H,

where $H \cdot \omega = d\theta$. Thus L is minimal

iff $\theta =$ constant iff L is special Lagrangian.

Write A for the second fundamental form of L,

$H = \mathrm{Trace}\, A$.

Now let L be compact, and consider the mean curvature flow $\{L_t : t \in [0,T)\}$ with $L_0 = L$, $T \in (0,\infty]$ maximal existence time.

Theorem (a) L_t is Lagrangian for $t \in [0,T)$. [Smoczyk, Wang.]

(b) If L is Maslov zero / graded then L_t is Hamiltonian isotopic to L $\forall t$. [In weak sense if immersed.]

(c) If L is graded then L_t is graded $\forall t$, and $\mathrm{var}\, L_t$ is decreasing in t.

(d) If $T < \infty$ then $\lim_{t \to T_-} \|A_t\|_{C^0} = \infty$, A_t 2FF of L_t.

There exists at least one singular point $x \in M$ of the flow, i.e. $\lim_{t \to T_-} \| A_t|_{L_t \cap U} \|_{C^0} = \infty$ for all open neighbourhoods U of x in M.

Finite time singularities of LMCF.

We are interested in the question:
if $\{L_t : t \in [0,T)\}$ is an LMCF with $T<\infty$, and $x \in M$ is a singular point of the flow at $t=T$, what might the flow look like near (x,T)?

Theorem (Neves 2010) Let L be a compact Lagrangian in a Calabi-Yau m-fold, $m \geqslant 2$. Then there exists $\tilde{L}$ Hamiltonian isotopic to L such that LMCF $\{\tilde{L}_t : t \in [0,T)\}$ starting from $\tilde{L}_0 = \tilde{L}$ develops a finite time singularity. Can make $T>0$ arbitrarily small. Neves' examples have $\mathrm{var}\, \tilde{L} > \pi$.

5 Thus, finite time singularities are common.

Type I and Type II singularities.

A MCF $\{L_t : t \in [0,T)\}$ with finite time singularity at $t=T$ is called Type I if $\|A_t\|^2_{C^0} \leq \dfrac{C}{T-t}$, some $C>0$, all $t \in [0,T)$.

Otherwise it is called Type II.

Type I singularities are quickly forming,
Type II singularities are slowly forming.

A Type I singularity admits a Type I blow up which is a MCF shrinker, and gives a good local model for the singularity.

Type II singularities are more difficult to describe.

6 They do admit "Type II blowups", but may not give a good picture.

Theorem (Wang, Chen-Li, Neves).
LMCF starting from a Maslov zero Lagrangian L (or graded, or almost calibrated) cannot develop a Type I singularity.
There do not exist graded LMCF shrinkers which could be Type I blowups of LMCF singularities.

So, all singularities of graded LMCF are Type II.
This makes them difficult to describe.

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Type II Blow-ups

Let $\{L_t : t \in [0,T)\}$ be an LMCF with a finite time singularity at $x \in M$, $t = T$.
Then we can do a Type II blow up. That is, we can find sequences $(x_i)_{i=1}^{\infty}$ in M, $(t_i)_{i=1}^{\infty}$ in $[0,T)$, $(\lambda_i)_{i=1}^{\infty}$ in $(0,\infty)$ with $x_i \to x$, $t_i \to T$, $\lambda_i \to \infty$ as $i \to \infty$, such that

$$\tilde{L}_s = \lim_{i \to \infty} \lambda_i \left(L_{t_i + \lambda_i^{-2} s} - x_i \right)$$

exists in $T_x M \cong \mathbb{C}^m$ for $s \in \mathbb{R}$, as a closed, nonempty Lagrangian in $\mathbb{C}^m$ with nonzero, bounded 2FF, and $\{\tilde{L}_s : s \in \mathbb{R}\}$ satisfies LMCF (an eternal solution).

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What are the possible Type II blow ups?

There are two obvious classes of Type II blow ups:

(A) $\tilde{L}_s = \tilde{L}$ is a <u>special Lagrangian</u> in $\mathbb{C}^m$,

(B) $\tilde{L}_s = \tilde{L} + sv$ for $\tilde{L}$ an <u>LMCF translator</u>, v translation vector in $\mathbb{C}^m$.

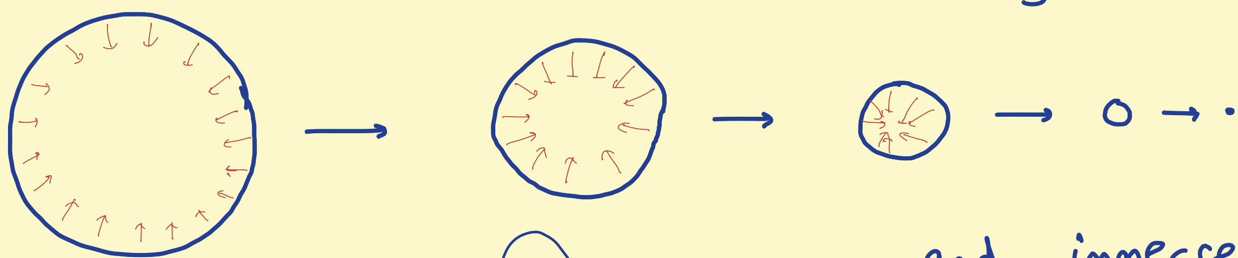
There are further constraints on $\tilde{L}$: it must have Euclidean $O(r^m)$ volume growth, and (loosely) be modelled on a special Lagrangian cone (or union of special Lagrangian cones) at infinity.

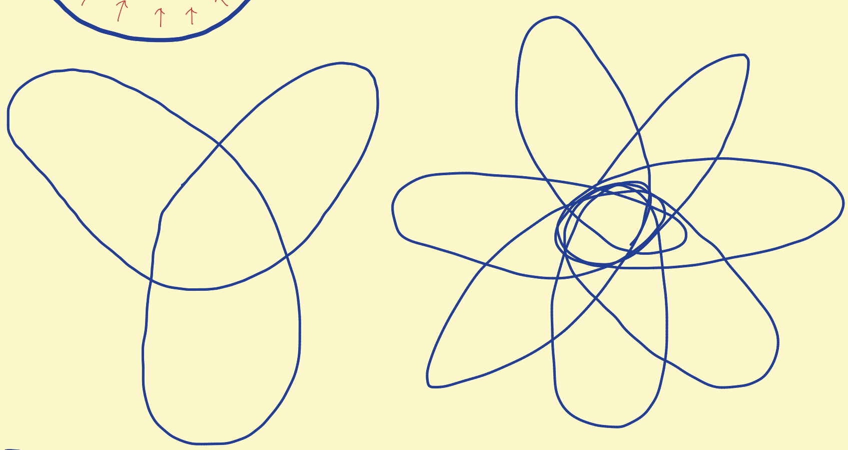
In both cases, $\tilde{L}$ must be <u>exact</u>.

9 This rules out a lot of interesting examples in case (A).

Example: LMCF singularities in dimension 1.

In the non-graded case, we can have Type I singularities modelled on MCF shrinkers: a shrinking circle





and immersed shrinking rosettes.

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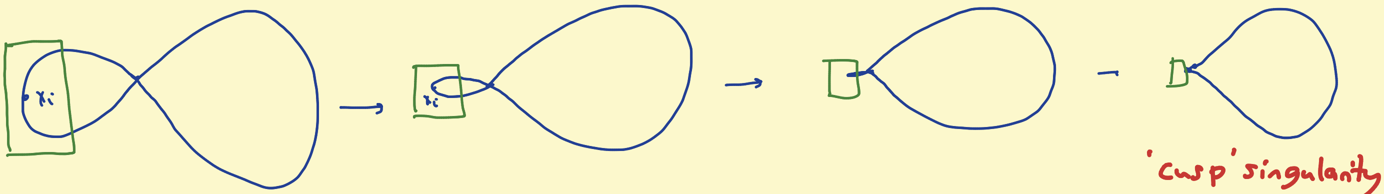
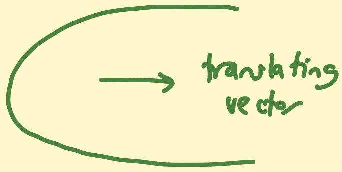
In the graded case, consider Lagrangian ∞ signs:

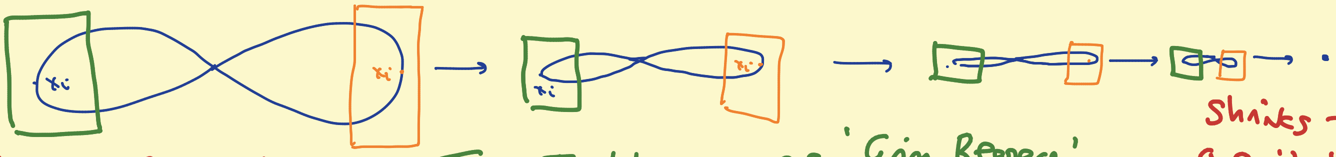


lobes unequal area.

Type II blowup is a 'Grim Reaper'



Note: * More than one Type II blowup.
* Type II blowup does not give a good global picture of the singularity.



lobes of equal area

Type II blowups are 'Grim Reapers'

Shrinks to a point in finite time, but not an LMCF shrinker.

LMCF singularities with special Lagrangian Type II blowup.

Suppose $\{L_t : t \in [0,T)\}$ has singularity at $x \in M$, $t = T$, and all Type II blowups at x are special Lagrangian.

General principles. * for $T-t$ small, L_t near x approximates an exact special Lagrangian $\tilde{L}_t$ in $T_xM = \mathbb{C}^m$, where $\tilde{L}_t$ should be asymptotic at ∞ to a SL cone C in $\mathbb{C}^m$.

* The flow converges quickly to approximately special Lagrangian near x. Then it moves slowly in the moduli space of exact AC special Lagrangians in $T_xM = \mathbb{C}^m$ with cone C until it hits a singular special Lagrangian.

* The motion in the moduli space of exact SL's with cone C is driven by 'outside influences', coming from the global geometry of L_t, away from x.

Fixing a special Lagrangian cone C in $\mathbb{C}^m$

This enables us to make some predictions. Fix a special Lagrangian cone C in $\mathbb{C}^m$. We'll consider LMCF singularities with SL Type II blow ups Σ asymptotic to C at infinity in $\mathbb{C}^m$.

Easy case: C has multiplicity 1, and has an isolated singularity at 0. Then AC SL m-folds Σ with cone C are well understood.

Difficult cases: C has multiplicity >1, as might need Σ to be a branched cover of C; or C has non-isolated singularities. Then AC SL m-folds Σ with cone C are difficult to study.

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The easy case: C multiplicity 1 isolated singularity.

Write $\mathcal{M}_C$ for the moduli space of exact AC SL m-folds Σ in $\mathbb{C}^m$ with cone C, modulo translations in $\mathbb{C}^m$. Then (Marshall, Pacini) $\mathcal{M}_C$ is a smooth manifold, of dimension $\mathrm{ind}(C)$ depending on # ends of C, and spectrum of Δ on link of C.

Expect: $\mathrm{ind}(C)-1$ should be the codimension in all Lagrangians, of Lagrangians which develop finite time LMCF singularity with cone C.

So, generic singularities have $\mathrm{ind}(C)=1$.

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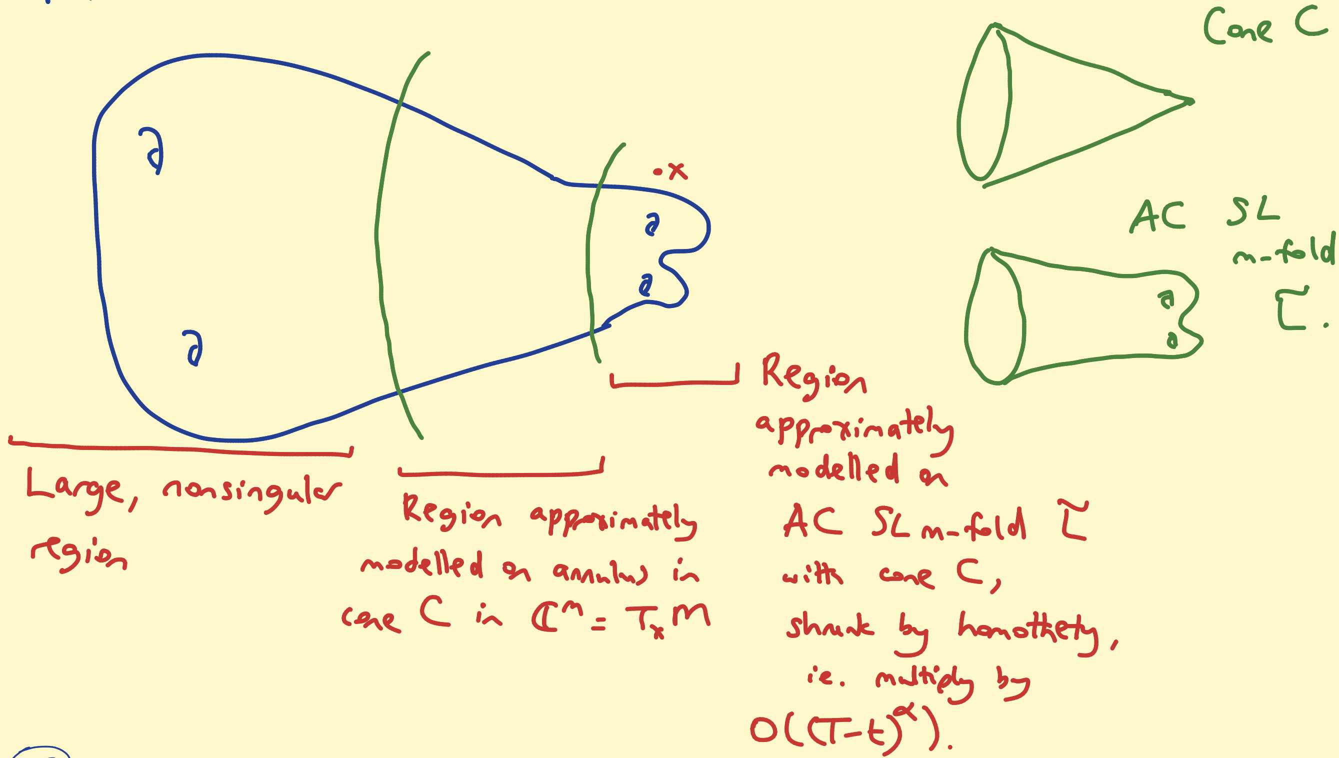
What L_t looks like in the easy case

For $T-t$ small, expect L_t to look like this:



An interesting analytical problem

<u>Problem:</u> Model LMCF analytically over short time periods, for Lagrangians L_t divided into 3 regions:

* "big region" * "annulus in cone C" region * "small AC SL" region

Explain how x, C, and the AC SL Σ_t evolve under the flow.

<u>Expect:</u> there is an "obstruction space" $O \cong \mathbb{R}^{\mathrm{ind}(C)}$ attached to C. Analysis on the "big region" determines an element of O, depending on t. This element of O determines how Σ_t evolves in M_C.

– Seems quite difficult.

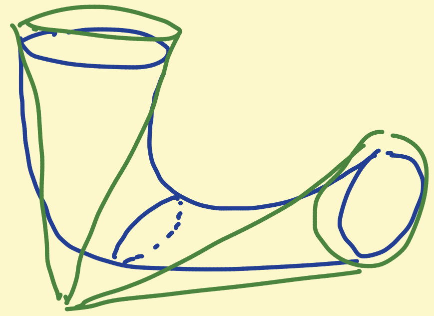
An important example: Lawlor necks

Take $C = \Pi_+ \cup \Pi_-$ to be the union of two special Lagrangian planes $\Pi_\pm$ in $\mathbb{C}^m$, intersecting transversely at $0 \in \mathbb{C}^m$, and satisfying an angle criterion.

Lawlor wrote down a family $\tilde{L}_A : A > 0$ of exact AC SL m-folds with cone C, all homothetic, diffeo to $S^{m-1} \times \mathbb{R}$.

Imagi-Joyce-Oliveira: any exact AC SL m-fold with cone C is a Lawlor neck.

Have $\operatorname{ind}(C) = 1$, the minimal value.



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Conjecture. The following 'neck pinch' behaviour is a generic singularity of LMCF in dimension $m \geq 2$ (that is, if it occurs for LMCF starting at L, it also occurs for LMCF starting from any small perturbation of L).

- We can have LMCF $\{L_t : t \in [0,T)\}$ such that $L_T = \lim_{t \to T} L_t$ is a nonsingular, immersed Lagrangian with a transverse self-intersection point at x. Two sheets L_T^+, L_T^- of L_T intersect at x, with the same phase. L_T has a different topology to L_t, $t < T$.

L_t looks like $L_T \,\#\,$(Lawlor neck). As $t \to T$, the Lawlor neck shrinks homothetically to $C = \Pi_+ \cup \Pi_-$.

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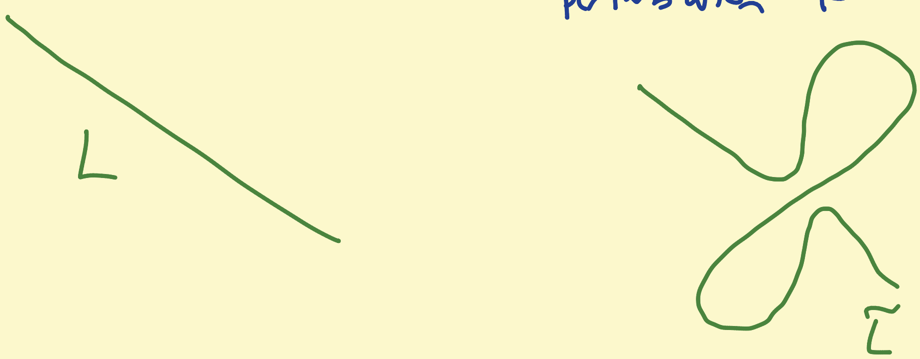
Relation to Neves' examples of finite time singularities.

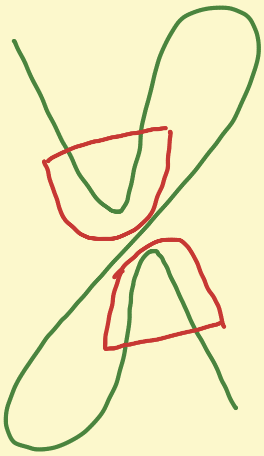
Given a Lagrangian L, Neves makes a Hamiltonian perturbation to $\tilde{L}$



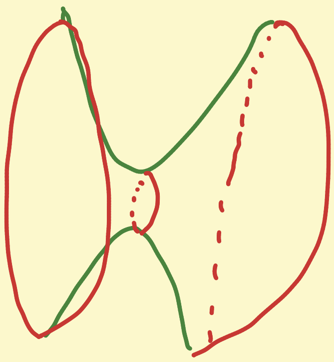
$SO(m)$-invariant picture; drawing is $L/SO(m)$, $\tilde{L}/SO(m)$.

and shows that LMCF starting from $\tilde{L}$ has a finite time singularity. He does not actually describe the singularity. I expect the first singularity is actually a 'neck pinch', as described in the conjecture.

Neves draws this picture, in $SO(m)$-invariant Lagrangians:



The red regions are part of a "small neck", $S^{m-1}\times\mathbb{R}$, like a Lawlor neck.



I expect this part of the diagram to shrink under LMCF, until it develops a neck pinch.

LMCF for $SO(m)$ invariant Lagrangians in $\mathbb{C}^m$

Joint work with Yng-Ing Lee.

Consider Lagrangians L_t in $\mathbb{C}^m$ of the form

$$L_t = \{ \lambda (x_1, \ldots, x_m) : (x_1, \ldots, x_m) \in \mathbb{R}^m, \; x_1^2 + \ldots + x_m^2 = 1, \; \lambda \in \gamma_t \subset \mathbb{C} \},$$

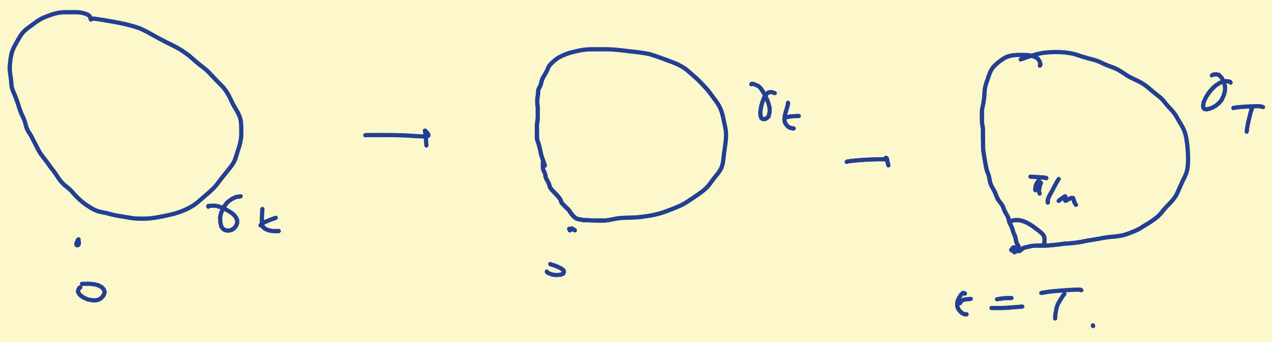
where $\gamma_t \subset \mathbb{C} \setminus 0$ is a closed curve, maybe immersed.

Then L_t is a Lagrangian $S^{m-1} \times S^1$ in $\mathbb{C}^m$, invariant under the action of $SO(m) \subset SU(m)$.

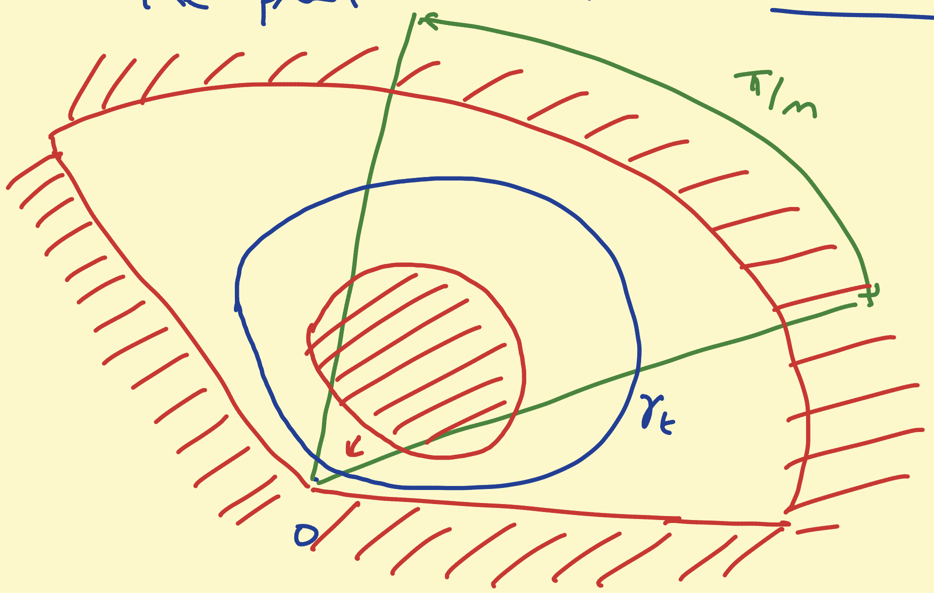
All $SO(m)$-invariant Lagrangians are of this form.

LMCF for such Lagrangians stays of such form; the curve γ_t evolves in $\mathbb{C}$, by a twisted version of curve-shortening flow.

Claim (DJ - Yng-Ing Lee) We can find a family of curves $\gamma_t : t \in [0, T)$ which evolve according to the $SO(m)$ invariant LMCF, and develop a singularity at $t = T$, such that the corresponding family $\{ L_t : t \in (0, T) \}$ undergoes a "neck pinch". At time T, γ_t falls into $0 \in \mathbb{C}$.



– The proof involves a barrier method:



we write down explicit red regions as shown satisfying a subharmonic (inequality) version of the flow. This means a curve γ_t satisfying the flow, not intersecting the red regions, never intersects them in future.

The red blob in the middle falls into 0 in finite time.

(23) The curve γ_t is trapped in between, and must converge to 0.

This actually gives us lots of examples of finite time singularities of LMCF:

If C is any special Lagrangian cone in $\mathbb{C}^m$ with isolated singularity at 0, so $\Sigma = C \cap S^{2m-1}$ is special Legendrian, compact, and nonsingular, then

$$L_t = \{ \lambda\sigma : \sigma \in \Sigma, \ \lambda \in \gamma_t \subset \mathbb{C} \}$$ for the same family of curves γ_t, satisfies LMCF with a finite time singularity at T. The cone is

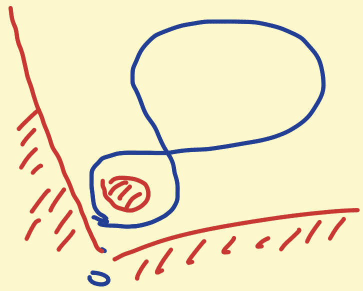
$C \cup e^{\pi i/m} C$, and the type II blow up is

(24) $\{ re^{i\theta}\sigma : \sigma \in \Sigma, \ \theta \in (0, \pi/m), \ r^m \sin m\theta = 1 \} \cong \Sigma \times \mathbb{R}$.

For general SL cones C, I expect there to be nongeneric singularities of LMCF, that occur in a finite codimension $\mathrm{ind}(C \cup e^{\pi i/m}C)$ amongst initial Lagrangians.

Anyway, this gives us an infinite number of local models for finite time singularities of LMCF, that do occur in compact LMCF in $\mathbb{C}^m$.

By taking γ_t to be a figure 8 can make L_t graded (though immersed).



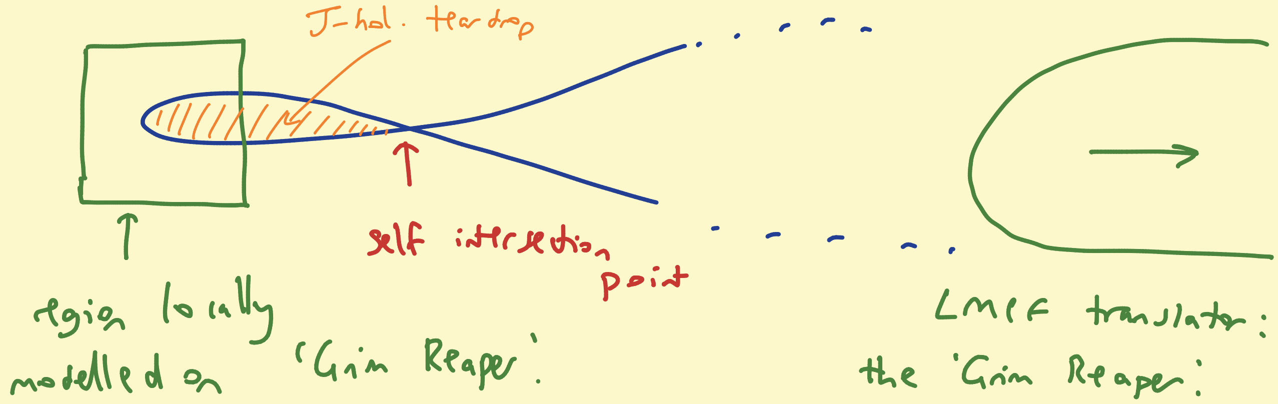
More exotic singularities of LMCF

We can also try and find examples of LMCF singularities whose Type II blow-ups are LMCF translators, or are SL m-fold singularities which are not isolated conical.

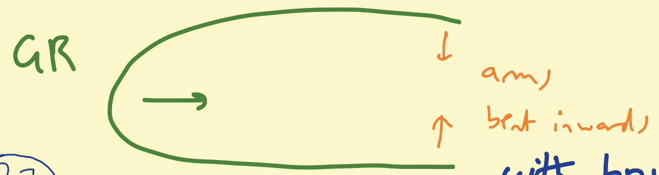
First consider the LMCF translator case.

<u>General principle</u>. Locally near x, the flow converges quickly to an approximate LMCF translator, then moves slowly in the moduli space of LMCF translators (this includes shrinking the translator) until it hits a singular LMCF translator. The motion in the moduli space of LMCF translators is driven by "outside influences" from the global geometry of L_t.

Go back to dimension $\underline{1}$ example:



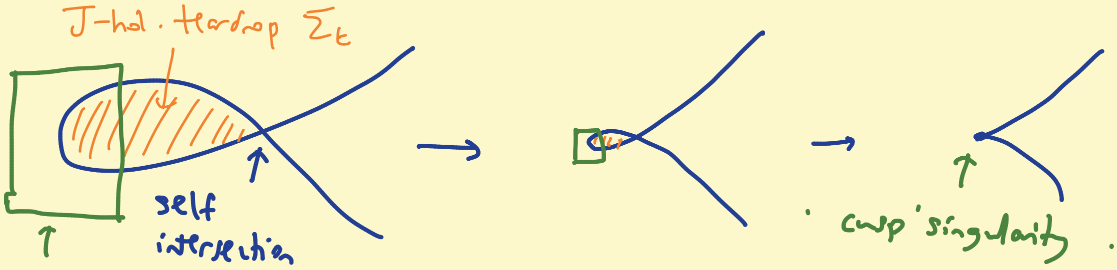
The "outside influence" which makes the Grim Reaper shrink, comes from the fact that the curve is bent round so it intersects itself.



Note that there is a J-holomorphic 'teardrop' with boundary in L; this obstructs HF^*.

Joyce-Lee-Tsui found an explicit family of LMCF translators in $\mathbb{C}^m$ for $m \geqslant 2$, generalizing the Grim Reaper in $\mathbb{C}$, diffeomorphic to $\mathbb{R}^m$.

Conjecture. There is a generic singularity of immersed LMCF in dimension $m \geqslant 2$, with Type II blow up a JLT translator. The Lagrangians L_t have a



self-intersection, and a J-holomorphic teardrop Σ_t with $\mathrm{area}(\Sigma_t) \to 0$. This Σ_t obstructs HF^* for L_t.

What about LMCF singularities modelled on SL singularities which are not isolated conical?

I studied SL 3-folds L in $\mathbb{C}^3$ invariant under the $U(1)$-action $e^{i\theta} : (z_1, z_2, z_3) \mapsto (e^{i\theta} z_1, e^{-i\theta} z_2, z_3)$.

Such L correspond to solutions of a nonlinear Cauchy-Riemann equation. I proved you can solve this on a disc, with prescribed boundary values, and get singular SL 3-folds.

These include isolated singularities with tangent cone $\mathbb{R}^3 \cup_{\mathbb{R}} \mathbb{R}^3$, which are not isolated conical.

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My results on $U(1)$ invariant SL 3-folds in $\mathbb{C}^3$ show that we can find a family $\tilde{L}_t$ of exact SL 3-folds in $\mathbb{C}^3$ near 0 such that:

* $\tilde{L}_t$ is nonsingular for $t < 0$
* $\tilde{L}_0$ has one singularity at 0, with tangent cone $\mathbb{R}^3 \cup_{\mathbb{R}} \mathbb{R}^3$
* $\tilde{L}_t$ has two singularities at $(0, 0, \pm\sqrt{t})$ for $t > 0$, each modelled on the SL T^2-cone

$$\{(re^{i\theta_1}, re^{i\theta_2}, re^{i\theta_3}) : r \geqslant 0, \ \theta_1 + \theta_2 + \theta_3 = 0\}.$$

This is a stable SL cone. Tapio Behrndt showed short-time existence for LMCF of Lagrangians with these kind of cone singularities.

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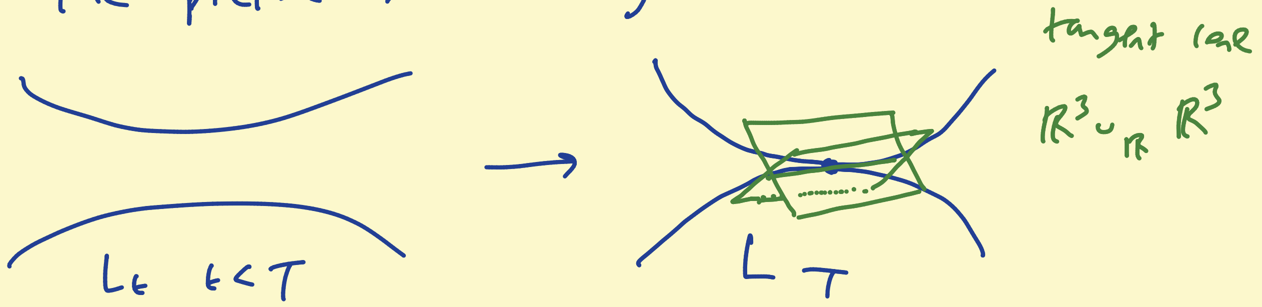
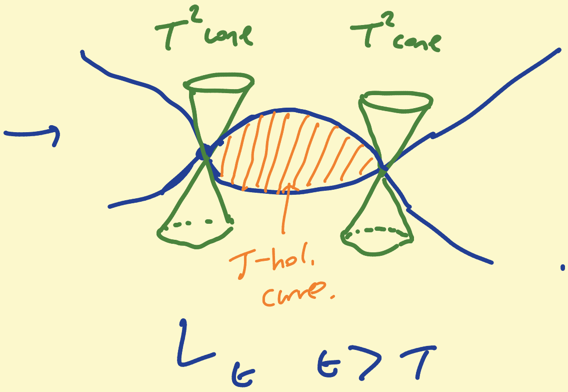
Conjecture. There is a generic singularity of LMCF in dimension $m=3$, with singular tangent cone $\mathbb{R}^3 \cup_{\mathbb{R}} \mathbb{R}^3$, which has non-isolated singularities. The local model for the formation of the singularity is like $\tilde{L}_t$, $t<0 \to \tilde{L}_0$ in the $U(1)$-invariant examples in the previous slide.

Furthermore, it should be possible to continue the LMCF for $t>T$, such that L_t for $t>T$ has two stable T^2-cone singularities, and the local model is like $\tilde{L}_t$, $t>0$ on the previous slide.

The picture is something like:





I expect this singularity of LMCF to be reversible: Two T^2 cone singularities in LMCF can come together and disappear, continuing the flow after a surgery.