Calibrated Geometry and Geometric Flows

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Lecture 1: Minimal submanifolds and introduction to calibrations

1 Minimal submanifolds

We start by analysing the submanifolds which are critical points for the volume functional, so-called minimal submanifolds.

1.1 First variation and definition

Let N be a submanifold (without boundary) of a Riemannian manifold (M, g) and let $F : N\times(-\epsilon, \epsilon) \to M$ be a variation of N with compact support; i.e. $F = \mathrm{Id}$ outside a compact subset $\overline{S}$ of N with S open and $F(p, 0) = p$ for all $p \in N$. The vector field $X = \frac{\partial F}{\partial t}|_N$ is called the variation vector field. We have the following definition.

Definition 1.1. N is *minimal* if $\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Vol}(F(S, t))|_{t=0} = 0$ for all variations F with compact support $\overline{S}$ (depending on F).

Remark Notice that we do not ask for N to minimize volume: it is only stationary for the volume.

Example. A plane in $\mathbb{R}^n$ is minimal since any small variation will have larger volume.

Example. Geodesics are locally length minimizing, so geodesics are minimal. However, as an example, the equator in $\mathcal{S}^2$ is minimal but not length minimizing since we can deform it to a shorter line of latitude.

For simplicity let us suppose that N is compact. We wish to calculate $\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Vol}(F(N, t))|_{t=0}$. Given local coordinates x_i on N we know that

$$\operatorname{Vol}(F(N,t)) = \int_N \sqrt{\det\left(g\left(\frac{\partial F}{\partial x_i}, \frac{\partial F}{\partial x_j}\right)\right)}\operatorname{vol}_N .$$

Let $p \in N$ and choose our coordinates x_i to be normal coordinates at p: i.e. so that $\frac{\partial F}{\partial x_i}(p, t) = e_i(t)$ satisfy $g(e_i(0), e_j(0)) = \delta_{ij}$. If $g_{ij}(t) = g(e_i(t), e_j(t))$ then we know that

$$\frac{\mathrm{d}}{\mathrm{d}t}\sqrt{\det(g_{ij}(t))}|_{t=0} = \frac{1}{2}\frac{\sum_i g'_{ii}(t)}{\sqrt{\det(g_{ij}(t))}}|_{t=0} = \frac{1}{2}\sum_i g'_{ii}(0).$$

Now

$$\begin{aligned}\frac{1}{2}\sum_i g'_{ii}(0) &= \frac{1}{2}\sum_i \frac{\mathrm{d}}{\mathrm{d}t} g\left(\frac{\partial F}{\partial x_i}, \frac{\partial F}{\partial x_i}\right)|_{t=0}\\ &= \sum_i g(\nabla_X e_i, e_i)\\ &= \sum_i g(\nabla_{e_i} X, e_i) = \operatorname{div}_N(X)\end{aligned}$$

since $[X, e_i] = 0$ (i.e. the t and x_i derivatives commute). Moreover, we see that

$$\operatorname{div}_N(X) = \sum_i g(\nabla_{e_i}X, e_i) = \operatorname{div}_N X^{\mathrm{T}} - \sum_i g(X^\perp, \nabla_{e_i}e_i) = \operatorname{div}_N X^{\mathrm{T}} - g(X, H)$$

(since $\nabla_{e_i} g(X^\perp, e_i) = 0$) where $^{\mathrm{T}}$ and $^\perp$ denote the tangential and normal parts and

$$H = \sum_i \nabla^\perp_{e_i} e_i$$

is the *mean curvature vector*. Overall we have the following.

Theorem 1.2. *The first variation formula is*

$$\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Vol}(F(N,t))|_{t=0} = \int_N \operatorname{div}_N(X)\operatorname{vol}_N = -\int_N g(X,H)\operatorname{vol}_N .$$

We deduce the following.

Definition 1.3. N is a *minimal submanifold* if and only if $H = 0$.

This is a *second order nonlinear PDE* (and in general a system, since H is a normal vector).

1.2 Graphs and examples

For a function $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ we will see that

$$N = \operatorname{Graph}(f) = \{(x, f(x)) \in \mathbb{R}^{n+1} \ : \ x \in U\}$$

is minimal if and only if

$$\operatorname{div}\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right) = 0.$$

We see that we can write this equation as $\Delta f + Q(\nabla f, \nabla^2 f) = 0$ where Q consists of nonlinear terms (but linear in $\nabla^2 f$). Hence, if we linearise this equation we just get $\Delta f = 0$, so f is harmonic. More concretely, linearising the operator $Pf = 0$ (at 0) means calculating the linear operator

$$Lf = L_0 Pf = \frac{\partial}{\partial t} P(tf)|_{t=0}.$$

In other words, the minimal submanifold equation is a nonlinear equation whose linearisation is just Laplace's equation: this is an example of a nonlinear *elliptic* PDE, which we shall discuss further later. For now, to compute the symbol of a linear operator L of order k, you compute

$$\sigma_L(x,\xi) = \lim_{t\to\infty} t^{-k} e^{-itf} L(e^{itf})(x)$$

where $\xi = \mathrm{d}f(x) \in T^*_x M$. Ellipticity says σ_L is an isomorphism whenever $\xi \neq 0$. In the case of the Laplacian $L = \Delta$ we get

$$\sigma_\Delta(x,\xi) = -|\xi|^2,$$

which is clearly an isomorphism for $\xi \neq 0$.

Example. A plane in $\mathbb{R}^n$ is trivially minimal because if X, Y are any vector fields on the plane then $\nabla^\perp_X Y = 0$ as the second fundamental form of a plane is zero.

Example. For curves γ, $H = 0$ is equivalent to the geodesic equation $\nabla_{\dot\gamma}\dot\gamma = 0$.

The most studied minimal submanifolds (other than geodesics) are minimal surfaces in $\mathbb{R}^3$, since here the equation $H = 0$ becomes a scalar equation on a surface, which is the simplest to analyse. In general we would have a system of equations, which is more difficult to study.

Example. The helicoid $M = \{(t\cos s, t\sin s, s) \in \mathbb{R}^3 \ : \ s, t \in \mathbb{R}\}$ is a complete embedded minimal surface, discovered by Meusnier in 1776.

Example. The catenoid $M = \{(\cosh t\cos s, \cosh t\sin s, t) \in \mathbb{R}^3 \ : \ s, t, \in \mathbb{R}\}$ is a complete embedded minimal surface, discovered by Euler in 1744 and shown to be minimal by Meusnier in 1776.

In fact the helicoid and the catenoid are locally isometric, and there is a 1-parameter family of locally isometric minimal surfaces deforming between the catenoid and helicoid.

It took about 70 years to find the next minimal surface, but now we know many examples of minimal surfaces in $\mathbb{R}^3$, as well in other spaces by studying the nonlinear elliptic PDE given by the minimal surface equation. The amount of literature in the area is vast, with key results including the Lawson and Willmore Conjectures, and minimal surfaces have applications to major problems in geometry including the Positive Mass Theorem, the Penrose Conjecture and the Poincaré Conjecture.

1.3 Volume minimization

Now, minimal submanifolds are only critical points for the volume functional, so a natural question to ask is: when is a minimal submanifold a minimizer for the volume functional?

This is very difficult to answer in general, and we see already for example that a plane is a minimizer but the catenoid is not a minimizer (simply by dilating it). We now see that minimal graphs are always volume minimizers, but even in this simple case the reason why we know that is due to calibrated geometry.

Theorem 1.4. *Suppose that $N = Graph(f)$ for $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$, where U is an open subset with compact closure. Let N' be a variation of N in $U \times \mathbb{R}$ with the same boundary as N. Then*

$$\mathrm{Vol}(N) \leq \mathrm{Vol}(N').$$

Proof. Since T^*N is trivial (as T^*U is trivial) we can choose a global orthonormal coframe $\xi_1, \ldots, \xi_n$ on N and form the n-form

$$\eta = \xi_1 \wedge \ldots \wedge \xi_n.$$

We can trivially extend η to $U \times \mathbb{R} \subseteq \mathbb{R}^{n+1}$ in a parallel fashion so that it is independent of the "vertical" x_{n+1} coordinate.

One fact is clear about η:

$$\eta(e_1, \ldots, e_n) \leq 1 \text{ for all unit tangent vectors } e_1, \ldots, e_n \text{ on } \mathbb{R}^{n+1}$$

since $\xi_1, \ldots, \xi_n$ are all unit.

The second fact is less clear: if N is minimal then

$$\mathrm{d}\eta = 0.$$

The reason is that $\mathrm{d}\eta$ is, up to a sign,

$$\mathrm{div}\left(\frac{\nabla f}{\sqrt{1 + |\nabla f|^2}}\right) \mathrm{vol}_{\mathbb{R}^{n+1}}$$

which vanishes precisely when N is minimal.

If N' is another submanifold in $U \times \mathbb{R}$ with the same boundary as N which is a variation of N, then there exists K compact and $n+1$-dimensional interpolating between N and N' and we can apply Stokes' Theorem:

$$0 = \int_K \mathrm{d}\eta = \int_{N'} \eta - \int_N \eta.$$

Why is this good? Well, $\eta|_N = \mathrm{vol}_N$ and $\eta|_{N'} \leq \mathrm{vol}_{N'}$ so

$$\mathrm{Vol}(N) = \int_N \mathrm{vol}_N = \int_N \eta = \int_{N'} \eta \leq \int_{N'} \mathrm{vol}_{N'} = \mathrm{Vol}(N').$$

Hence N is volume-minimizing. □

Another question to ask is: how do we find minimal submanifolds? There are two main answers to this. The first, and most natural, is the variational approach. That is, simply minimize the volume functional. The problem with this is that the topological space of submanifolds does not have good compactness properties: i.e. it is quite easy to find a sequence of compact submanifolds with a uniform bound on their volume whose limit is not a smooth submanifold. Therefore, one has to enlarge the space of submanifolds to a weaker notion, for example integral varifolds or integral currents. We will discuss this briefly later. The point is that if you directly minimize you have no control on the minimizing sequence and you might end up with some very singular object at the end. The task is then to prove that maybe it is not as singular as you thought, and in the best case scenario that is smooth. This is precisely the method by which Hodge theory is proved. However, this is much more challenging in the case of submanifolds, and very often one has to deal with complicated singularities.

1.4 Mean curvature flow

We see in the first variation formula

$$\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Vol}(F(N,t))|_{t=0} = -\int_N g(X,H)\operatorname{vol}_N$$

that we make this most negative when we choose X to be a multiple of H. This leads us to the following.

Definition 1.5. If we define a variation satisfying

$$\frac{\partial F}{\partial t} = H,$$

then this would be the *negative gradient flow* of the volume functional. The variation satisfying this equation is called *mean curvature flow* (MCF). We often write the flow as (N_t) where $N_t = F(N,t)$ and $N_0 = N$.

MCF represents the fastest way to decrease volume and is an example of a geometric flow. Thus the second approach to finding minimal submanifolds would be to start with an arbitrary submanifold and then try to solve MCF. The problem then is to understand the long-time behaviour of the flow.

Typically, MCF becomes singular in finite time.

Example. If you take a round sphere $N = \mathcal{S}^n(1)$ of radius 1 in $\mathbb{R}^{n+1}$ then the solution of MCF is $\mathcal{S}^n(r(t))$ where $r(t) \to 0$ as $t \to T < \infty$, where r is an explicit function of t which you should compute. This example shows that N_t is just a rescaling of N_0 which is getting smaller as t increases: such a solution to MCF is called a *self-shrinker*.

Example. A fundamental result of Huisken says that a compact convex hypersurface in $\mathbb{R}^{n+1}$ will always develop a finite-time singularity under MCF where it shrinks to a point p. Moreover, if you rescale both space and time around p, you will see a round sphere. Thus all compact convex hypersurfaces are diffeomorphic to $\mathcal{S}^n$.

The previous example suggests that we can use MCF to study the interaction of Riemannian geometry with topology. This is in fact the case.

We will not dwell on standard MCF in this course (there will be full course on this next year by Felix Schulze), but there are two cases we want to note which are interesting.

Example. We can write the mean curvature flow for graphs as

$$\frac{\partial f}{\partial t} = \sqrt{1+|\nabla f|^2}\,\operatorname{div}\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right).$$

The linearisation of this equation is

$$\frac{\partial f}{\partial t} = \Delta f,$$

i.e. the heat equation.

It essentially follows from the previous example that MCF is a nonlinear *parabolic* PDE, meaning it is of the form $\frac{\partial f}{\partial t} = Pf$ where P is a (negative) nonlinear elliptic operator. In general, again it is a system of PDE which is hard to solve.

Example. For curves in the plane $\mathbb{R}^2$, mean curvature flow is often called curve shortening flow and is written

$$\frac{\partial F}{\partial t} = \kappa$$

where κ is the curvature of the curve.

We will see that there are natural geometric flows which are related to calibrated geometry.

2 Introduction to calibrated geometry

As we have seen, minimal submanifolds are extremely important. However there are two key issues.

- Minimal submanifolds are defined by a second order nonlinear PDE system – therefore they are hard to analyse.
- Minimal submanifolds are only critical points for the volume functional, but we are often interested in minima for the volume functional – we need a way to determine when this occurs.

We can help resolve these issues using the notion of calibration, introduced by Harvey–Lawson (1982).

Definition 2.1. A differential k-form η on a Riemannian manifold (M, g) is a *calibration* if

- $\mathrm{d}\eta = 0$ and
- $\eta(e_1, \ldots, e_k) \leq 1$ for all unit tangent vectors $e_1, \ldots, e_k$ on M.

Example. We see that the form $\eta = \xi_1 \wedge \ldots \wedge \xi_n$ we define on $U \times \mathbb{R}$ from a minimal graph is a calibration on $U \times \mathbb{R}$.

Example. Any form with constant coefficients on $\mathbb{R}^n$ can be rescaled so that it is a calibration with at least one plane where equality holds.

This example shows that there are many calibrations η, but the interesting question is: for which planes $V = \mathrm{Span}\{e_1, \ldots, e_k\}$ does $\eta(e_1, \ldots, e_k) = 1$? More importantly, can we find submanifolds N so that this equality holds on each tangent space? This motivates the next definition.

Definition 2.2. Let η be a calibration k-form on (M, g). An oriented k-dimensional submanifold N of (M, g) is *calibrated* by η if $\eta|_N = \mathrm{vol}_N$, i.e. if for all $p \in N$ we have $\eta(e_1, \ldots, e_k) = 1$ for an oriented orthonormal basis $e_1, \ldots, e_k$ for T_pN.

Example. Any plane in $\mathbb{R}^n$ is calibrated. If we change coordinates so that the plane P is $\{x \in \mathbb{R}^n : x_{k+1} = \ldots = x_n = 0\}$ then $\eta = \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_k$ is a calibration and P is calibrated by η.

Example. A minimal graph is calibrated.

Notice that the calibrated condition is now an algebraic condition on the tangent vectors to N, so being calibrated is a *first order nonlinear PDE*. We shall motivate these definitions further later, but for now we make the following observation.

Theorem 2.3. *Let N be a calibrated submanifold. Then N is minimal and moreover if F is any variation with compact support $\overline{S} \subseteq N$ then* $\mathrm{Vol}(F(S,t)) \geq \mathrm{Vol}(S)$*; i.e. N is volume-minimizing.*

Proof. Suppose that N is calibrated by η and suppose for simplicity that N is compact. We will show that N is homologically volume-minimizing.

Suppose that N' is homologous to N. Then,

$$\mathrm{Vol}(N) = \int_N \eta = \int_{N'} \eta \leq \mathrm{Vol}(N')$$

by Stokes' Theorem as $\mathrm{d}\eta = 0$, since because N, N' are homologous there exists a compact manifold K with boundary $-N \cup N'$ and by Stokes' Theorem

$$0 = \int_K \mathrm{d}\eta = \int_{N'} \eta - \int_N \eta.$$

We have the result by the definition of minimal submanifold, since N is a critical point for the volume functional. □

We conclude this introduction with the following elementary result.

Proposition 2.4. *There are no compact calibrated submanifolds in $\mathbb{R}^n$.*

Proof. Suppose that η is a calibration and N is compact and calibrated by η. Then $\mathrm{d}\eta = 0$ so by the Poincaré Lemma $\eta = \mathrm{d}\zeta$, and hence

$$\mathrm{Vol}(N) = \int_N \eta = \int_N \mathrm{d}\zeta = 0$$

by Stokes' Theorem. □

Although there are many calibrations, having calibrated submanifolds greatly restricts the calibrations you want to consider. The calibrations which have calibrated submanifolds have special significance and there is a particular connection with special holonomy, due to the following observations.

Let G be a holonomy group of a Riemannian metric g on an n-manifold M: that is, the group generated by parallel transport around loops in M. Then G acts on the k-forms on $\mathbb{R}^n$, so suppose that η_0 is a G-invariant k-form. We can always rescale η_0 so that $\eta_0|_P \leq \mathrm{vol}_P$ for all oriented k-planes P and equality holds for at least one P. Since η_0 is G-invariant, if P is calibrated then so is $\gamma \cdot P$ for any $\gamma \in G$, which usually means we have quite a few calibrated planes. We know by the *holonomy principle* that we then get a parallel k-form η on M which is identified with η_0 at every point. Since $\nabla\eta = 0$, we have $\mathrm{d}\eta = 0$ and hence η is a calibration. Moreover, we have a lot of calibrated tangent planes on M, so we can hope to find calibrated submanifolds.

Lecture 2: Complex and special Lagrangian submanifolds

3 Complex submanifolds

We would now like to address the question: where does the calibration condition come from? The answer is from *complex geometry.* On $\mathbb{R}^{2n} = \mathbb{C}^n$ with coordinates $z_j = x_j + iy_j$, we have the complex structure J and the distinguished Kähler 2-form (a closed, in fact parallel, positive $(1,1)$-form)

$$\omega = \sum_{j=1}^{n} \mathrm{d}x_j \wedge \mathrm{d}y_j = \frac{i}{2}\sum_{j=1}^{n} \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j.$$

More generally we can work with a *Kähler manifold* (M, J, ω). Our first key result is the following.

Theorem 3.1. *On a Kähler manifold (M, J, ω), $\frac{\omega^k}{k!}$ is a calibration whose calibrated submanifolds are the complex k-dimensional submanifolds: i.e. submanifolds N such that $J(T_pN) = T_pN$ for all $p \in N$.*

Since $\mathrm{d}\omega^k = k\mathrm{d}\omega \wedge \omega^{k-1} = 0$, the theorem follows immediately from the following result.

Theorem 3.2 (Wirtinger's inequality)**.** *For any unit vectors $e_1, \ldots, e_{2k} \in \mathbb{C}^n$,*

$$\frac{\omega^k}{k!}(e_1, \ldots, e_{2k}) \leq 1$$

with equality if and only if $\mathrm{Span}\{e_1, \ldots, e_{2k}\}$ *is a complex k-plane in $\mathbb{C}^n$.*

Before proving this we make the following observation, which we leave as an exercise.

Lemma 3.3. *If η is a calibration and $*\eta$ is closed then $*\eta$ is a calibration.*

Proof. We see that $|\frac{\omega^k}{k!}|^2 = \frac{n!}{k!(n-k)!}$ and $\mathrm{vol}_{\mathbb{C}^n} = \frac{\omega^n}{n!}$ so $*\frac{\omega^k}{k!} = \frac{\omega^{n-k}}{(n-k)!}$. Hence, by the lemma, it is enough to study the case where $k \leq \frac{n}{2}$.

Let P be any $2k$-plane in $\mathbb{C}^n$ with $2k \leq n$. We shall find a canonical form for P. First consider $\langle Ju, v\rangle$ for orthonormal unit vectors $u, v \in P$. This must have a maximum, so let $\cos\theta_1 = \langle Ju, v\rangle$ be this maximum where $0 \leq \theta_1 \leq \frac{\pi}{2}$.

Suppose that $w \in P$ is a unit vector orthogonal to $\mathrm{Span}\, u, v$. The function

$$f_w(\theta) = \langle Ju, \cos\theta v + \sin\theta w\rangle$$

has a maximum at $\theta = 0$ so $f_w'(0) = \langle Ju, w\rangle = 0$. Similarly we have that $\langle Jv, w\rangle = 0$, and thus $w \in \mathrm{Span}\{u, v, Ju, Jv\}^{\perp}$.

We then have two cases. If $\theta_1 = 0$ then $v = Ju$ so we can set $u = e_1, v = Je_1$ and see that $P = \mathrm{Span}\{e_1, Je_1\} \times Q$ where Q is a $2(k-1)$-plane in $\mathbb{C}^{n-1} = \mathrm{Span}\{e_1, Je_1\}^{\perp}$. If $\theta_1 \neq 0$ we have that $v = \cos\theta_1 Ju + \sin\theta_1 w$ where w is a unit vector orthogonal to u and Ju, so we can let $u = e_1$, $w = e_2$ and see that $P = \mathrm{Span}\{e_1, \cos\theta_1 Je_1 + \sin\theta_1 e_2\} \times Q$ where Q is a $2(k-1)$-plane in $\mathbb{C}^{n-2} = \mathrm{Span}\{e_1, Je_1, e_2, Je_2\}^{\perp}$.

Proceeding by induction we see that we have an oriented basis $\{e_1, Je_1, \ldots, e_n, Je_n\}$ for $\mathbb{C}^n$ so that

$$P = \mathrm{Span}\{e_1, \cos\theta_1 Je_1 + \sin\theta_1 e_2, \ldots, e_{2k-1}, \cos\theta_k Je_{2k-1} + \sin\theta_k e_{2k}\},$$

where $0 \leq \theta_1 \leq \ldots \leq \theta_{k-1} \leq \frac{\pi}{2}$ and $\theta_{k-1} \leq \theta_k \leq \pi - \theta_{k-1}$.

Since we can write $\omega = \sum_{j=1}^{n} e^j \wedge Je^j$ we see that $\frac{\omega^k}{k!}$ restricts to P to give a product of $\cos\theta_j$ which is certainly less than or equal to 1. Moreover, equality holds if and only if all of the $\theta_j = 0$ which means that P is complex. ☐

Corollary 3.4. *Compact complex submanifolds of Kähler manifolds are homologically volume-minimizing.*

We know that complex submanifolds are defined by holomorphic functions; i.e. solutions to the Cauchy–Riemann equations, which are a first-order PDE system.

Example. $N = \{(z, \frac{1}{z}) \in \mathbb{C}^2 : z \in \mathbb{C} \setminus \{0\}\}$ is a complex curve in $\mathbb{C}^2$, and thus is calibrated.

Example. An important non-trivial example of a Kähler manifold is $\mathbb{CP}^n$, where the zero set of a system of polynomial equations defines a (singular) complex submanifold.

The previous example shows that calibrated submanifolds need not be smooth and that the singularities of calibrated submanifolds can be very complicated.

4 Special Lagrangians

Complex submanifolds are very familiar, but can we find any other interesting classes of calibrated submanifolds? The answer is that indeed we can, particularly when the manifold has special holonomy. We begin with the case of holonomy $\mathrm{SU}(n)$ – so-called *Calabi–Yau manifolds*. The model example for Calabi–Yau manifolds is $\mathbb{C}^n$ with complex structure J, Kähler form ω and holomorphic volume form

$$\Upsilon = \mathrm{d}z_1 \wedge \ldots \wedge \mathrm{d}z_n,$$

if $z_1, \ldots, z_n$ are complex coordinates on $\mathbb{C}^n$. By Yau's solution of the Calabi Conjecture, one can define compact Calabi–Yau n-folds as Kähler manifolds with vanishing first Chern class. This means that they have a nowhere vanishing holomorphic volume form and they have a Ricci-flat Kähler metric.

4.1 Definitions and examples

Theorem 4.1. *Let M be a Calabi–Yau manifold with holomorphic volume form Υ. Then $\mathrm{Re}(e^{-i\theta}\Upsilon)$ is a calibration for any $\theta \in \mathbb{R}$.*

Since $\mathrm{d}\Upsilon = 0$, the result follows immediately from the following result.

Theorem 4.2. *On $\mathbb{C}^n$, $|\Upsilon(e_1, \ldots, e_n)| \leq 1$ for all unit vectors $e_1, \ldots, e_n$ with equality if and only if $P = \mathrm{Span}\{e_1, \ldots, e_n\}$ is a Lagrangian plane, i.e. P is an n-plane such that $\omega|_P \equiv 0$.*

Proof. Let $e_1, \ldots, e_n$ be the standard basis for $\mathbb{R}^n$ and let P be an n-plane in $\mathbb{C}^n$. There exists $A \in \mathrm{GL}(n, \mathbb{C})$ so that $f_1 = Ae_1, \ldots, f_n = Ae_n$ is an orthonormal basis for P. Then $\Upsilon(Ae_1, \ldots, Ae_n) = \det_{\mathbb{C}}(A)$ so

$$|\Upsilon(f_1, \ldots, f_n)|^2 = |\det{}_{\mathbb{C}}(A)|^2 = |\det{}_{\mathbb{R}}(A)| = |f_1 \wedge Jf_1 \wedge \ldots \wedge f_n \wedge Jf_n| \leq |f_1||Jf_1| \ldots |f_n||Jf_n| = 1$$

with equality if and only if $f_1, Jf_1, \ldots, f_n, Jf_n$ are orthonormal. However, this is exactly equivalent to the Lagrangian condition, since $\omega(u, v) = g(Ju, v)$ so $\omega|_P \equiv 0$ if and only if $JP = P^\perp$. □

Definition 4.3. Notice that this theorem implies that if N is an oriented Lagrangian submanifold of $\mathbb{C}^n$, or more generally of a Calabi–Yau, then

$$\Upsilon|_N = e^{i\theta}\,\mathrm{vol}_N$$

for some function $e^{i\theta} : N \to \mathcal{S}^1$. This function $e^{i\theta}$ is called the phase and θ the Lagrangian angle of N. Notice that θ is not well-defined (it is only defined up to multiples of 2π), but that $\mathrm{d}\theta$ is. The cohomology class (up to a sign and dividing by 2π) $[\mathrm{d}\theta] \in H^1(N)$ is called the *Maslov class* of the Lagrangian N.

Example. Let

$$L = \{\frac{1}{\sqrt{2}}(e^{i\theta_1}, e^{i\theta_2}) \in \mathbb{C}^2 \,:\, \theta_1, \theta_2 \in \mathbb{R}\}.$$

Then you should find that the Lagrangian angle is given by $\theta = \theta_1 + \theta_2 \pmod{\pi}$. This shows that in this case that the Lagrangian angle is not single-valued and so in this case the Maslov class of L is not zero. This example is the well-known Clifford torus in $\mathcal{S}^3$, made famous by the Willmore and Lawson Conjectures. It is minimal in $\mathcal{S}^3$ but certainly not minimal in $\mathbb{C}^2$.

Definition 4.4. A submanifold N of M calibrated by $\mathrm{Re}(e^{-i\theta}\Upsilon)$ is called *special Lagrangian* with phase $e^{i\theta}$. If $\theta = 0$ we say that N is simply special Lagrangian. By the previous theorem, we see that N is special Lagrangian if and only if $\omega|_N \equiv 0$ (i.e. N is Lagrangian) and $\mathrm{Im}\,\Upsilon|_N \equiv 0$ (up to a choice of orientation so that $\mathrm{Re}\,\Upsilon|_N > 0$).

In other words, special Lagrangians are Lagrangians with constant Lagrangian angle θ, and are therefore zero-Maslov, i.e. their Maslov class vanishes.

Example. Consider $\mathbb{C} = \mathbb{R}^2$ with coordinates $z = x + iy$, complex structure J given by $Jw = iw$, Kähler form $\omega = \mathrm{d}x \wedge \mathrm{d}y = \frac{i}{2}\mathrm{d}z \wedge \mathrm{d}\bar{z}$ and holomorphic volume form $\Upsilon = \mathrm{d}z = \mathrm{d}x + i\mathrm{d}y$. We want to consider the special Lagrangians in $\mathbb{C}$, which are 1-dimensional submanifolds or curves N in $\mathbb{C} = \mathbb{R}^2$.

Since ω is a 2-form, it vanishes on any curve in $\mathbb{C}$. Hence every curve is $\mathbb{C}$ is Lagrangian. For N to be special Lagrangian with phase $e^{i\theta}$ we need that

$$\mathrm{Re}(e^{-i\theta}\Upsilon) = \cos\theta\mathrm{d}x + \sin\theta\mathrm{d}y$$

is the volume form on N, or equivalently that

$$\mathrm{Im}(e^{-i\theta}\Upsilon) = \cos\theta\mathrm{d}y - \sin\theta\mathrm{d}x$$

vanishes on N. This means that $\cos\theta\partial_x + \sin\theta\partial_y$ is everywhere a unit tangent vector to N, so N is a straight line given by $N = \{(t\cos\theta, t\sin\theta) \in \mathbb{R}^2 : t \in \mathbb{R}\}$ (up to translation), so it makes an angle θ with the x-axis, hence motivating the term "phase $e^{i\theta}$".

Notice that this result is compatible with the fact that special Lagrangians are minimal, and hence must be geodesics in $\mathbb{R}^2$; i.e. straight lines.

The previous example immediately generalises to say the torus T^2 with is standard Calabi–Yau structure, where all curves are Lagrangian and geodesics are special Lagrangian for some phase.

Example. Consider $\mathbb{C}^2 = \mathbb{R}^4$. We know that $\omega = \mathrm{d}x_1 \wedge \mathrm{d}y_1 + \mathrm{d}x_2 \wedge \mathrm{d}y_2$. We also know that $\mathrm{Re}\,\Upsilon = \mathrm{d}x_1 \wedge \mathrm{d}x_2 + \mathrm{d}y_2 \wedge \mathrm{d}y_1$, which looks somewhat similar. In fact, if we let J' denote the complex structure given by $J'(\partial_{x_1}) = \partial_{x_2}$ and $J'(\partial_{y_2}) = \partial_{y_1}$, then $\mathrm{Re}\,\Upsilon = \omega'$, the Kähler form corresponding to the complex structure J'. Hence special Lagrangians in $\mathbb{C}^2$ are complex curves for a different complex structure.

In fact, we have a hyperkähler triple of complex structures J_1, J_2, J_3, where $J_1 = J$ is the standard one and $J_3 = J_1J_2 = -J_2J_1$ so that $J_1 = J_2J_3 = -J_3J_2$ and $J_2 = J_3J_1 = -J_1J_3$, and the corresponding Kähler forms are $\omega = \omega_1$, ω_2, ω_3 which are orthogonal and the same length with $\Upsilon = \omega_2 + i\omega_3$.

This shows we should only consider complex dimension 3 and higher to find new calibrated submanifolds.

Example. $\mathrm{SU}(n)$ acts transitively on the space of special Lagrangian planes with isotropy $\mathrm{SO}(n)$. So any special Lagrangian plane is given by $A \cdot \mathbb{R}^n$ for $A \in \mathrm{SU}(n)$ where $\mathbb{R}^n$ is the standard real $\mathbb{R}^n$ in $\mathbb{C}^n$.

Given $\theta = (\theta_1, \ldots, \theta_n)$ we can define a plane $P(\theta) = \{(e^{i\theta_1}x_1, \ldots, e^{i\theta_n}x_n) \in \mathbb{C}^n : (x_1, \ldots, x_n) \in \mathbb{R}^n\}$ (where we can swap orientation). We see that $P(\theta)$ is is special Lagrangian if and only if $\mathrm{Re}\,\Upsilon|_P = \pm\cos(\theta_1 + \ldots + \theta_n) = 1$ so that $\theta_1 + \ldots + \theta_n \in \pi\mathbb{Z}$. Given any $\theta_1, \ldots, \theta_n \in (0, \pi)$ with $\theta_1 + \ldots + \theta_n = \pi$, there exists a special Lagrangian N (called a *Lawlor neck*) asymptotic to $P(0) \cup P(\theta)$. It is diffeomorphic to $\mathcal{S}^{n-1} \times \mathbb{R}$. By rotating coordinates we have a special Lagrangian with phase i asymptotic to $P(-\frac{\theta}{2}) \cup P(\frac{\theta}{2})$.

The simplest case is when $\theta_1 = \ldots = \theta_n = \frac{\pi}{n}$: here N is called the *Lagrangian catenoid*. When $n = 2$, under a coordinate change the Lagrangian catenoid becomes the complex curve $\{(z, \frac{1}{z}) \in \mathbb{C}^2 : z \in \mathbb{C}\backslash\{0\}\}$ that we saw before. When $n = 3$, the only possibilities for the angles are $\sum_i \theta_i = \pi, 2\pi$, but if $\sum_i \theta_i = 2\pi$ we can rotate coordinates and change the order of the planes so that $P(0) \cup P(\theta)$ becomes $P(0) \cup P(\theta')$ where $\sum_i \theta'_i = \pi$. Hence, given any pair of transverse special Lagrangian planes in $\mathbb{C}^3$, there exists a Lawlor neck asymptotic to their union.

We can find special Lagrangians in Calabi–Yaus using the following easy result.

Proposition 4.5. *Given a Calabi–Yau manifold (M, ω, Υ) and $\sigma : M \to M$ be such that $\sigma^2 = \mathrm{Id}$, $\sigma^*(\omega) = -\omega$, $\sigma^*(\Upsilon) = \overline{\Upsilon}$. Then Fix$(\sigma)$ is special Lagrangian, if it is non-empty.*

Example. Let $X = \{[z_0, \ldots, z_4] \in \mathbb{CP}^4 : z_0^5 + \ldots + z_4^5 = 0\}$ (the *Fermat quintic*) with its Calabi–Yau structure (which exists by Yau's solution of the Calabi conjecture since the first Chern class of X vanishes). Let σ be the restriction of complex conjugation on $\mathbb{CP}^4$ to X. Then the fixed point set of σ, which is the real locus in X, is a special Lagrangian 3-fold (if it is non-empty). (There is a subtlety here: σ is certainly an anti-holomorphic isometric involution for the induced metric on X, but this is *not* the same as the Calabi–Yau metric on X. Nevertheless, it is the case that σ satisfies the conditions of the proposition above.)

Example. There exists a Calabi–Yau metric on $T^*\mathcal{S}^n$ (the Stenzel metric) so that the base $\mathcal{S}^n$ is special Lagrangian. When $n = 2$ this is a hyperkähler metric called the Eguchi–Hanson metric.

4.2 The PDE and relation to minimal Lagrangians

We now go back to graphs to give us a sense of what is going on in general. Let $N = \mathrm{Graph}(F)$ where $F : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ and we view $\mathbb{R}^n \times \mathbb{R}^n = \mathbb{C}^n$ with the first $\mathbb{R}^n$ as the "real" $\mathbb{R}^n$. Then when is N Lagrangian? Well, if we choose complex coordinates $x_j + iy_j$ as before then consider the 1-form

$$\tau = \sum_j y_j \mathrm{d}x_j.$$

This form has the nice property that $\mathrm{d}\tau = -\omega$. Moreover, if we write F in components as $F = (F_1, \ldots, F_n)$ then

$$\tau|_N = \sum_j F_j \mathrm{d}x_j = F$$

if we view F as the 1-form $\sum_j F_j \mathrm{d}x_j$. Hence,

$$\mathrm{d}\tau|_N = \sum_{j,k} \frac{\partial F_j}{\partial x_k} \mathrm{d}x_k \wedge \mathrm{d}x_j = \mathrm{d}F.$$

Hence, N is Lagrangian (so $\omega|_N = 0$) if and only if $\mathrm{d}F = 0$, i.e. F is a closed 1-form. What about the special Lagrangian condition? Well, $\Upsilon = \mathrm{d}z_1 \wedge \ldots \wedge \mathrm{d}z_n$ so

$$\Upsilon|_N = \det(I + i\nabla F)$$

where ∇F is the matrix of partial derivatives of F. So, N is special Lagrangian if and only if

$$\operatorname{Im}\det(I + i\nabla F) = 0.$$

This is a fully nonlinear first order elliptic PDE. Since F is closed, it is locally exact so we can write $F = \mathrm{d}f$ for a function $f : \mathbb{R}^n \to R$, which then means that the condition above becomes

$$Pf = \operatorname{Im}\det(I + i\mathrm{Hess}f) = 0$$

where $\mathrm{Hess}f$ is the Hessian matrix of second derivatives of f.

If $n = 1$, this equation just says $\frac{\partial^2 f}{\partial x^2} = 0$ which defines a straight line, as we know. If $n = 2$ we see that

$$\operatorname{Im}\det \begin{pmatrix} 1 + if_{11} & if_{12} \\ if_{12} & 1 + if_{22} \end{pmatrix} = f_{11} + f_{22} = -\Delta f.$$

If $n = 3$ then we have

$$\operatorname{Im}\det \begin{pmatrix} 1 + if_{11} & if_{12} & if_{31} \\ if_{12} & 1 + if_{22} & if_{23} \\ if_{31} & if_{23} & 1 + if_{33} \end{pmatrix} = f_{11} + f_{22} + f_{33} - \det(\mathrm{Hess}f),$$

so the special Lagrangian equation is

$$-\Delta f = \det(\mathrm{Hess}f) = MA(f),$$

the real Monge–Ampère operator acting on f.

In general, if we diagonalise $\mathrm{Hess}f$ with eigenvalues $\lambda_1, \ldots, \lambda_n$ then

$$\operatorname{Im}\det(I + i\mathrm{Hess}f) = \operatorname{Im}\Pi_j(1 + i\lambda_j) = \Pi_j\sqrt{1 + \lambda_j^2}\sin(\sum_j \tan^{-1}\lambda_j) = 0,$$

where the Lagrangian angle is

$$\theta = \sum_j \tan^{-1}\lambda_j$$

(up to multiples of 2π, since we need $\operatorname{Re}\Upsilon|_N$ to be positive on N). Hence the special Lagrangian equation can be written

$$Pf = \sum_j \tan^{-1}\lambda_j = 0,$$

which, unlike the minimal submanifold equation, is *fully nonlinear* in f. The linearisation therefore is

$$Lf = \lambda_1 + \ldots + \lambda_n = \operatorname{tr} \operatorname{Hess} f = -\Delta f.$$

Therefore the special Lagrangian equation is elliptic.

We claimed that calibrated submanifolds are minimal so special Lagrangians are necessarily minimal, but is it possible to see this directly? This answer comes from the following result.

Theorem 4.6. *Let L be an oriented Lagrangian in a Calabi–Yau manifold and let θ be the Lagrangian angle of L. Then*

$$H = J\nabla\theta.$$

Hence L is minimal if and only if it is special Lagrangian (with some phase).

Proof. Let $p \in L$ and let $e_1, \ldots, e_n$ be a local orthonormal frame on L near p, which means $e_1, \ldots, e_n$, $Je_1, \ldots, Je_n$ is a local orthonormal basis for T_qM for all q in some open neighbourhood p in L. Suppose further that we have chosen geodesic normal coordinates at p which means that $(\nabla_{e_j} e_k)^{\mathrm{T}} = 0$ at p. Hence $\nabla_{e_j} e_k$ lies in $(T_pL)^\perp = \operatorname{Span}\{Je_1, \ldots, Je_n\}$ by the Lagrangian condition.

Now we know that

$$\Upsilon(e_1, \ldots, e_n) = e^{i\theta} \operatorname{vol}_N(e_1, \ldots, e_n) = e^{i\theta}.$$

Therefore,

$$\nabla_{e_j}\left(\Upsilon(e_1, \ldots, e_n)\right) = \nabla_{e_j}(e^{i\theta}) = (i\nabla_{e_j}\theta)e^{i\theta}.$$

We can also do the calculation another way. Since Υ is parallel we have $\nabla_{e_j}\Upsilon = 0$. Therefore, at p,

$$\begin{aligned}
\nabla_{e_j}\left(\Upsilon(e_1, \ldots, e_n)\right) &= \sum_k \Upsilon(e_1, \ldots, \nabla_{e_j} e_k, \ldots, e_n) \\
&= \sum_{k,l} g(\nabla_{e_j} e_k, Je_l)\Upsilon(e_1, \ldots, Je_l, \ldots, e_n) \\
&= \sum_{k,l} ig(\nabla_{e_j} e_k, Je_l)\Upsilon(e_1, \ldots, e_l, \ldots, e_n)
\end{aligned}$$

since Υ is of type $(n, 0)$. We deduce that we must have $l = k$ to get something non-zero and so using that Je_k is normal and $[e_j, e_k] = 0$, together with $\nabla J = 0$, we see that

$$\begin{aligned}
\nabla_{e_j}\left(\Upsilon(e_1, \ldots, e_n)\right) &= \sum_k ig(\nabla_{e_j} e_k, Je_k)e^{i\theta} \\
&= \sum_k ig(\nabla_{e_k} e_j, Je_k)e^{i\theta} \\
&= -\sum_k ig(e_j, \nabla_{e_k} Je_k)e^{i\theta} \\
&= -\sum_k ig(e_j, J\nabla_{e_k} e_k)e^{i\theta} \\
&= -ig(e_j, JH)e^{i\theta}.
\end{aligned}$$

Comparing our expressions we see that

$$\nabla_{e_j}\theta = -g(e_j, JH)$$

and therefore

$$\nabla\theta = -JH.$$

Multiply both sides by J gives the result. □

I cannot overstate the importance of this equation. It implies that the only critical points of the volume functional restricted to the space of Lagrangians are absolute minima. Thus one might image one can contract the space of Lagrangians to the critical points, namely the special Lagrangians. This will clearly play a crucial role in understanding Lagrangian mean curvature flow.

Lecture 3: Properties of special Lagrangians and analytic methods

4.3 Moduli space and regularity

I now want to describe the moduli space of compact special Lagrangians.

Before doing so, I want to make a little aside concerning regularity of minimal (and hence) calibrated submanifolds. We shall recall a basic part of the theory of elliptic operators on compact manifolds N, namely *Schauder theory.*

If L is a linear elliptic operator of order l and we look at *Hölder spaces* $C^{k,a}$, then if $w \in C^{k,a}$ and $Lv = w$ then $v \in C^{k+l,a}$ and there is a universal constant C so that

$$\|v\|_{C^{k+l,a}} \leq C(\|Lv\|_{C^{k,a}} + \|v\|_{C^0})$$

(and we can drop the $\|v\|_{C^0}$ term if v is orthogonal to $\operatorname{Ker} L$, which is always finite-dimensional). There is a slightly improved version of this where L is not a smooth operator but has coefficients of regularity $C^{k,a}$.

Suppose we are on a compact manifold N and we want to solve $P(f) = 0$ where

$$P(f) = -\operatorname{div}\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right)$$

is the minimal hypersurface operator on functions f. Let us consider regularity for f. We cannot just apply elliptic regularity as it stands because P is nonlinear. However, since $P(f)$ is quasilinear (i.e. linear in the second derivatives of f) we can re-arrange $P(f) = 0$ by taking all of the second derivatives to one side as:

$$R(x, \nabla f(x))\nabla^2 f(x) = E(x, \nabla f(x))$$

where $x \in N$ and $R, E : C^{k+1,a} \to C^{k,a}$. Since $L_0 P = \Delta$ is elliptic and ellipticity is an open condition we know that the operator L_f (depending on f) given by

$$L_f(h)(x) = R(x, \nabla f(x))\nabla^2 h(x)$$

is a *linear* second order elliptic operator whenever $\|\nabla f\|_{C^0}$ is small, in particular if $\|f\|_{C^{1,a}}$ is sufficiently small (in fact, it is *always* elliptic as long as ∇f is bounded). The operator L_f does not have smooth coefficients, but if $f \in C^{k,a}$ then the coefficients $R \in C^{k-1,a}$.

Suppose that $f \in C^{2,a}$ and $\|f\|_{C^{1,a}}$ is small with $P(f) = 0$. Then $L_f(f) = E(f)$ and L_f is a linear *second order* elliptic operator with coefficients in $C^{1,a}$ and $E(f)$ in $C^{1,a}$. So by elliptic regularity we can deduce that $f \in C^{3,a}$. We have gained one degree of regularity, so we can "bootstrap", i.e. proceed by induction and deduce that any $C^{2,a}$ solution to $P(f) = 0$ is smooth. In fact, as long as we can make sense of $L_f(h)$ where $h \in C^{1,a}$ (which we can do by integration by parts), then in fact any $C^{1,a}$ solution to $P(f) = 0$ will be smooth by the same argument.

Example. $C^{1,a}$-minimal submanifolds (and thus calibrated submanifolds) are *smooth.* The reason why we can reduce to this graph case is that any $C^{1,a}$-minimal submanifold N near any given point $p \in N$ will be a minimal graph over T_pN.

You might ask how we say that a $C^{1,a}$ submanifold is minimal, given that we cannot differentiate twice. The reason is the first variation formula:

$$\frac{\partial}{\partial t}\operatorname{Vol}(F(N,t))|_{t=0} = \int_N \operatorname{div}_N X \operatorname{vol}_N$$

where X is the variation vector field. The right-hand side makes sense as long as you can differentiate once on N, and so it makes sense for C^1-submanifolds. In fact, it even makes to say that it vanishes for so-called varifolds, which are even weaker notions of submanifolds that we may well discuss later. One can think of varifolds as the objects for which the first variation formula makes sense (hence the name).

Remark More sophisticated techniques can be used to deduce that C^1-minimal submanifolds are real analytic. Notice that elliptic regularity results are *not* valid for C^k spaces, so this result is not obvious.

We now want to study a moduli space problem. What does this mean? It simply means: how many solutions to our nonlinear PDE are there near a given solution?

Suppose we stick with our simple equation $P(f) = 0$ and suppose we want to describe $P^{-1}(0)$ for f near 0. Then we can re-arrange $P(f) = 0$ using the linearisation $L = \Delta$ of P at 0 as

$$\Delta f + Q(\nabla f, \nabla^2 f) = 0,$$

where Q is nonlinear but linear in $\nabla^2 f$. We know, on a compact manifold, that

$$\int_N \Delta f \,\mathrm{vol}_N = 0.$$

Moreover, by the Fredholm alternative, we know that we can solve $\Delta f = g$ if and only if $\int_N g\,\mathrm{vol}_N = 0$. Therefore, we can solve $P(f) = 0$ if and only if

$$\int_N Q(\nabla f, \nabla^2 f)\,\mathrm{vol}_N = \int_N P(f)\,\mathrm{vol}_N = 0.$$

Of course, we know this is the case since $P(f)$ is the divergence of something. In other words we know can solve $\Delta f_0 = -Q(\nabla f, \nabla^2 f)$, which means that $P(f) \in \operatorname{Im} L$ for all f. This means the linearisation is surjective onto a space containing the image of P, which means we are in the setting for implementing the Implicit Function Theorem for Banach spaces to conclude that we can always solve $P(f) = 0$ for some f near 0 (which in this case will be unique up to constants) and f will be smooth by our regularity argument above. In general, we will use the following.

Theorem 4.7 (Implicit Function Theorem)**.** *Let X, Y be Banach spaces, let $U \ni 0$ be open in X, let $P : U \to Y$ with $P(0) = 0$ and $L_0P : X \to Y$ surjective with finite-dimensional kernel K.*

Then for some U, $P^{-1}(0) = \{u \in U : P(u) = 0\}$ is a manifold of dimension $\dim K$. Moreover, if we write $X = K \oplus Z$, $P^{-1}(0) = \operatorname{Graph} G$ for some map G from an open set in K to Z with $G(0) = 0$.

This gives us a way to describe all perturbations of a given calibrated submanifold, as we now see in the special Lagrangian case.

Theorem 4.8 (McLean)**.** *Let N be a compact special Lagrangian in a Calabi–Yau manifold M. Then the moduli space of deformations of N is a smooth manifold of dimension $b^1(N)$.*

Remark One should compare this result to the deformation theory for complex submanifolds in Kähler manifolds. There, one does not get that the moduli space is a smooth manifold: in fact, it can be singular, and one has *obstructions* to deformations. It is somewhat remarkable that special Lagrangian calibrated geometry enjoys a much better deformation theory than this classical calibrated geometry.

Proof. The tubular neighbourhood theorem gives us a diffeomorphism $\exp : S \subseteq \nu(N) \to T \subseteq M$ which maps the zero section to N acting as the identity; in other words, we can write any nearby submanifold to N as the graph of a normal vector field on N. We know that N is Lagrangian, so the complex structure J gives an isomorphism between $\nu(N)$ and TN and the metric gives an isomorphism between TN and T^*N: $v \mapsto g(Jv, .) = \omega(v, .) = \alpha_v$. Therefore any deformation of N in T is given as the graph of a 1-form. In fact, using the Lagrangian neighbourhood theorem, we can arrange that any $N' \in T$ is the graph of a 1-form α, so that if $f_\alpha : N \to N_\alpha$ is the natural diffeomorphism then

$$f_\alpha^*(\omega) = \mathrm{d}\alpha \quad \text{and} \quad -*f_\alpha^*(\operatorname{Im}\Upsilon) = F(\alpha, \nabla\alpha) = \mathrm{d}^*\alpha + Q(\alpha, \nabla\alpha).$$

Hence, N_α is special Lagrangian if and only if $P(\alpha) = (F(\alpha, \nabla\alpha), \mathrm{d}\alpha) = 0$. This means that infinitesimal special Lagrangian deformations are given by closed and coclosed 1-forms, which is the kernel of L_0P.

Since $\operatorname{Im}\Upsilon = 0$ on N we have that $[\operatorname{Im}\Upsilon] = 0$ on N_α, so

$$P : C^\infty(S) \to \mathrm{d}^*(C^\infty(T^*N)) \oplus \mathrm{d}(C^\infty(T^*N)) \subseteq C^\infty(\Lambda^0 T^*N \oplus \Lambda^2 T^*N).$$

If we let $X = C^{1,a}(T^*N)$, $Y = \mathrm{d}^*(C^{1,a}(T^*N)) \oplus \mathrm{d}(C^{1,a}(T^*N))$ and $U = C^{1,a}(S)$ we can apply the Implicit Function Theorem if we know that

$$L_0P : \alpha \in X \mapsto (\mathrm{d}^*\alpha, \mathrm{d}\alpha) \in Y$$

is surjective, i.e. given $\mathrm{d}\beta + \mathrm{d}^*\gamma \in Y$ does there exist α such that $\mathrm{d}\alpha = \mathrm{d}\beta$ and $\mathrm{d}^*\alpha = \mathrm{d}^*\gamma$? If we let $\alpha = \beta + \mathrm{d}f$ then we need $\Delta f = \mathrm{d}^*\mathrm{d}f = \mathrm{d}^*(\gamma - \beta)$. Since

$$\int_N \mathrm{d}^*(\gamma - \beta)\,\mathrm{vol}_N = \pm \int_N \mathrm{d} * (\gamma - \beta) = 0$$

we can solve the equation for f, and hence $L_0 P$ is surjective.

Therefore $P^{-1}(0)$ is a manifold of dimension $\dim \mathrm{Ker}\, L_0 P = b^1(N)$ by Hodge theory. Moreover, if $P(\alpha) = 0$ then N_α is special Lagrangian, hence minimal and since $\alpha \in C^{1,a}$ we deduce that α is in fact smooth. □

Example. The special Lagrangian $\mathcal{S}^n$ in $T^*\mathcal{S}^n$ has $b^1 = 0$ and so is rigid.

Observe that if we have a special Lagrangian T^n in M then $b^1(T^n) = n$ and its deformations locally foliate M, so we can hope to find special Lagrangian torus fibrations. This cannot happen in compact manifolds without singular fibres, but still motivates the SYZ conjecture in Mirror Symmetry. The deformation result also motivates the following theorem.

Theorem 4.9 (Bryant). *Every compact oriented real analytic Riemannian 3-manifold can be isometrically embedded in a Calabi–Yau 3-fold as the fixed point set of an involution.*

The reason why we restrict to compact oriented 3-dimensional manifolds N is that they are parallelizable; i.e. TN (and hence T^*N) is trivial, so we can try to build a Calabi–Yau metric on a neigbourhood of the zero section in $T^*N = N \times \mathbb{R}^3$.

4.4 Uniqueness

I would like to make a brief interlude to discussion the question of uniqueness. It follows from the McLean theorem that we proved that if we take a special Lagrangian homology sphere then it is rigid and so locally unique. It was proposed (by Joyce) that one might want to “count” special Lagrangian homology 3-spheres to define a new invariant(s) of Calabi–Yau 3-folds. However, for this to be remotely plausible you would like to have a more global uniqueness statement rather than just local uniqueness.

But uniqueness in what class? We have seen that special Lagrangians can have a nontrivial moduli space, if $b^1 > 0$, so we need to restrict the class of deformations we want to consider. The answer is to restrict to the so-called *Hamiltonian isotopy class* of a Lagrangian N. This means the set of Lagrangians N' so that there exists a smooth family of Lagrangians submanifolds $f_t : N \to M$ with $f_0 = \mathrm{id}$, $f_1(N) = N'$ and a family h_t of functions (called Hamiltonians) on N with

$$\frac{\partial f_t}{\partial t} = J\nabla h_t.$$

The corresponding 1-form deforming N at $t = 0$ to first order would be $J\nabla h_0 \lrcorner \omega = -dh_0$, which is exact, so since the tangent space to the moduli space of special Lagrangians is given by the cohomology class of the special Lagrangian we see that locally any compact special Lagrangian is unique in its Hamiltonian isotopy class.

Can we make this a global statement? The answer is yes, but not without a clever argument and some big fancy machinery. If the Hamiltonian functions h_t were independent of t, then the proof is “easy” in the sense that it is essentially the maximum principle for elliptic equations. However, if not there is no analytic way currently to prove it.

The full statement involves saying something about Lagrangian Floer homology, so I won’t state it, but the simple version is the following.

Theorem 4.10 (Thomas–Yau). *Let N be a special Lagrangian which is spin and has $b^2(N) = 0$ (in particular, a homology n-sphere for $n \geq 3$). Then N is unique in its Hamiltonian isotopy class.*

The proof is clever and uses a Morse theory argument but relies on the unobstructedness of the Lagrangian Floer homology of N, so I will not discuss the proof.

You can also ask whether one can prove any kind of uniqueness in the non-compact setting, say in $\mathbb{C}^n$. In $\mathbb{C}^n$ one has a certain 1-form

$$\lambda = \sum_j x_j \mathrm{d}y_j - y_j \mathrm{d}x_j$$

which satisfies $\mathrm{d}\lambda = 2\omega$. Hence λ is closed on any Lagrangian N. The easiest way for it to be closed is if it is exact, which leads to the following.

Definition 4.11. We say that a Lagrangian N in $\mathbb{C}^n$ is *exact* if $\lambda|_N$ is exact.

Example. Any Lagrangian sphere is exact (but a Lagrangian sphere cannot be embedded).

We can now state an important uniqueness result.

Theorem 4.12 (Imagi–Joyce–Oliveira dos Santos)**.** *Let $P(\theta) \cup P(0)$ be a union of a pair of tranverse special Lagrangian planes with $\sum_j \theta_j = \pi$ and let N be an exact special Lagrangian n-fold for $n \geq 3$ which is asymptotic to this pair of planes. Then N is a Lawlor neck.*

The proof is a much more sophisticated version of the Thomas–Yau uniqueness argument. The statment is also known for $n = 2$ via complex geometry (since recall that special Lagrangian surfaces are complex curves for a different complex structure).

We have a related uniqueness result by myself and André Neves for $n = 2$ and for Imagi–Joyce–Oliveira dos Santos for $n \geq 3$.

Theorem 4.13 (L.–Neves/Imagi–Joyce–Oliveira dos Santos)**.** *Let $P(\theta) \cup P(0)$ be a union of a pair of transverse special Lagrangian planes with $\sum_j \theta_j < \pi$. Then there is a unique exact Lagrangian self-expander N asymptotic to this pair of planes; i.e. N evolves under mean curvature flow purely by an expanding dilation so the solution to the flow is $N_t = \sqrt{t}N$ for $t > 0$.*

Interestingly, the proof by L.–Neves only works for $n = 2$ (and gives local uniqueness for $n \geq 3$) using analytic methods, whereas the other proof only works for $n \geq 3$ and uses Lagrangian Floer homology again. The second proof cannot be extended to $n = 2$ but potentially the analytic proof could be extended to $n \geq 3$ if one understood singularities better. In general, it would be interesting to know whether analytic methods can prove the known uniqueness statements using the Fukaya category methods.

4.5 Singularities

I now want to make a brief interlude discussing singularities of special Lagrangians.

Even very singular special Lagrangians have a lot of structure. We know that at almost every point it has a tangent plane and that at singular points it will have a tangent cone (though it is a major open question whether this tangent cone is unique). Hence we can understand singularities of special Lagrangians by understanding special Lagrangian cones.

Example. The only special Lagrangian cones in $\mathbb{C}^2$ are planes. The reason is that the link of the cone will be a curve in $\mathcal{S}^3$ which is minimal, so it must be a great circle and hence the cone is a plane.

Example. A pair of (transverse) planes is the simplest possible singularity model we can get. This happens when we have an immersed special Lagrangian with a transverse self-intersection point. We know that such singularities are closely related to Lawlor necks.

Example. Bryant showed that the only special Lagrangian cones in $\mathbb{C}^3$ whose link is $\mathcal{S}^2$ is a plane.

Example. There is a very important special Lagrangian cone in $\mathbb{C}^3$ whose link is T^2, due to Harvey–Lawson:

$$C = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \, : \, |z_1|^2 = |z_2|^2 = |z_3|^2, \, \mathrm{Re}(z_1 z_2 z_3) \geq 0, \, \mathrm{Im}(z_1 z_2 z_3) = 0\}.$$

This is invariant under the maximal torus T^2 in SU(3):

$$(e^{i\theta_1}, e^{i\theta_2}) \cdot (z_1, z_2, z_3) = (e^{i\theta_1} z_1, e^{i\theta_2} z_2, e^{-i(\theta_1+\theta_2)} z_3).$$

Hence the link of C is T^2. One can compute that it is special Lagrangian. There are three different special Lagrangian smoothings known of C, all T^2-invariant and constructed by Harvey–Lawson, for example:

$$N = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \,:\, |z_1|^2 = |z_2|^2 = |z_3|^2 - 1,\, \mathrm{Re}(z_1 z_2 z_3) \geq 0,\, \mathrm{Im}(z_1 z_2 z_3) = 0\}.$$

It is still unknown whether C can appear as a tangent cone to a compact special Lagrangian 3-fold in a Calabi–Yau, but there has been recent work of Imagi which proves that C can occur for special Lagrangians but his example is not in the Calabi–Yau setting.

Example. Haskins–Kapouleas proved there are infinitely many special Lagrangian cones in $\mathbb{C}^3$ whose link can have arbitrarily high genus. It is unknown whether these cones can be smoothed or not, and we do not know whether they can appear as tangent cones for compact special Lagrangians.

In higher dimensions special Lagrangian cones become extremely complicated, and so it seems that the task of studying singularities even for special Lagrangian 3-folds seems very challenging.

4.6 Gluing

Another well-known way to get a solution of a linear PDE from two solutions is simply to add them. However, for a nonlinear PDE $P(v) = 0$ this will not work. Intuitively, we can try to add two solutions to give us a solution v_0 for which $P(v_0)$ is small. Then we may try to perturb v_0 by v to solve $P(v+v_0) = 0$.

Geometrically, this occurs when we have two calibrated submanifolds N_1, N_2 and then glue them together to give a submanifold N which is "almost" calibrated, then we deform N to become calibrated. If the two submanifolds N_1, N_2 are glued using a very long neck then one can imagine that N is almost the disjoint union of N_1, N_2 and so close to being calibrated. If instead one scales N_2 by a factor t and then glues it into a singular point of N_1, we can again imagine that as t becomes very small N resembles N_1 and so again is close to being calibrated. These two examples are in fact related, because if we rescale the shrinking N_2 to fixed size, then we get a long neck between N_1 and N_2 of length of order $-\log t$. However, although these pictures are appealing, they also reveal the difficulty in this approach: as t becomes small, N becomes more "degenerate", giving rise to analytic difficulties which are encoded in the geometry of N_1, N_2 and N.

These ideas are used extensively in geometry, and particularly successfully in calibrated geometry (e.g. Haskins–Kapouleas, Joyce, Y.-I. Lee, L-, Pacini). A particular simple case is the following, which we will describe to show the basic idea of the gluing method.

Theorem 4.14. *Let N be a compact connected 3-manifold and let $i : N \to M$ be a special Lagrangian immersion with tranverse self-intersection points in a Calabi–Yau manifold M. Then there exist embedded special Lagrangians N_t such that $N_t \to N$ as $t \to 0$.*

Remark One might ask about the sense of convergence here: for definiteness, we can say that N_t converges to N in the sense of currents; that is, if we have any compactly supported 3-form χ on M then $\int_{N_t} \chi \to \int_N \chi$ as $t \to 0$. However, all sensible notions of convergence of submanifolds will be true in this setting.

Proof. At each self-intersection point of N the tangent spaces are a pair of transverse 3-planes, which we can view as a pair of tranverse special Lagrangian 3-planes P_1, P_2 in $\mathbb{C}^3$. Since we are in dimension 3, we know that there exists a (unique up to scale) special Lagrangian Lawlor neck L asymptotic to $P_1 \cup P_2$. We can then glue tL into N near each intersection point to get a compact embedded (if we glue in a Lawlor neck for every self-intersection point) submanifold $S_t = N\#tL$. We can also arrange that S_t is Lagrangian, i.e. that it is a Lagrangian connect sum.

Now we want to perturb S_t to be special Lagrangian. Since S_t is Lagrangian, by the deformation theory we can write any nearby submanifold as the graph of a 1-form α, and this graph will be special Lagrangian if and only if (using the same notation as in our deformation theory discussion)

$$P_t(\alpha) = (- * f_\alpha^*(\mathrm{Im}\,\Upsilon), f_\alpha^*(\omega)) = 0.$$

Since S_t is Lagrangian but not special Lagrangian we have that

$$f_\alpha^*(\omega) = \mathrm{d}\alpha \quad \text{and} \quad - *f_\alpha^*(\mathrm{Im}\,\Upsilon) = P_t(0) + \mathrm{d}_t^*\alpha + Q_t(\alpha, \nabla\alpha)$$

where $P_t(0) = -*\mathrm{Im}\Upsilon|_{S_t}$ and $\mathrm{d}_t^* = L_0 P_t$, which is a perturbation of the usual d^* since we are no longer linearising at a point where $P_t(0) = 0$. By choosing $\alpha = \mathrm{d}f$, we then have to solve

$$\Delta_t f = -P_t(0) - Q_t(\nabla f, \nabla^2 f)$$

where Δ_t is a perturbation of the Laplacian.

For simplicity, let us suppose that Δ_t is the Laplacian on S_t. The idea is to view our equation as a fixed point problem. We know that if we let $X^k = \{f \in C^{k,a}(N) : \int_N f \,\mathrm{vol}_N = 0\}$ then $\Delta_t : X^{k+2} \to X^k$ is an isomorphism so it has an inverse G_t. We know by our elliptic regularity result that there exists a constant $C(\Delta_t)$ such that

$$\|f\|_{C^{k+2,a}} \leq C(\Delta_t)\|\Delta_t f\|_{C^{k,a}} \Leftrightarrow \|G_t h\|_{C^{k+2,a}} \leq C(\Delta_t)\|h\|_{C^{k,a}}$$

for any $f \in X^{k+2}$, $h \in X^k$.

We thus see that $P_t(f) = 0$ for $f \in X^{k+2}$ if and only if

$$f = G_t(-P_t(0) - Q_t(f)) = F_t(f).$$

The idea is now to show that F_t is a contraction sufficiently near 0 for all t small enough. Then it will have a (unique) fixed point near 0, which will also be smooth because it satisfies $P_t(f) = 0$ and hence defines a special Lagrangian as the graph of $\mathrm{d}f$ over S_t.

We know that $F_t : X^{k+2} \to X^{k+2}$ with

$$\|F_t(f_1) - F_t(f_2)\|_{C^{k+2,a}} = \|G_t(Q_t(f_1) - Q_t(f_2))\|_{C^{k+2,a}} \leq C(\Delta_t)\|Q_t(f_1) - Q_t(f_2)\|_{C^{k,a}}.$$

Since Q_t and its first derivatives vanish at 0 we know that

$$\|Q_t(f_1) - Q_t(f_2)\|_{C^{k,a}} \leq C(Q_t)\|f_1 - f_2\|_{C^{k+2,a}}(\|f_1\|_{C^{k+2,a}} + \|f_2\|_{C^{k+2,a}}).$$

Hence, F_t is a contraction on $\overline{B_{\epsilon_t}(0)} \subseteq X^{k+2}$ if we can choose ϵ_t so that

$$2C(\Delta_t)\|P_t(0)\|_{C^{k,a}} \leq \epsilon_t \leq \frac{1}{2C(\Delta_t)C(Q_t)}.$$

(This also proves our earlier Implicit Function Theorem result by hand since there $P_t(0) = P(0) = 0$ so we just need to take ϵ_t small enough.) In other words, we need that

- $P_t(0)$ is small, so S_t is "close" to being calibrated and is a good approximation to $P_t(f) = 0$;
- $C(\Delta_t), C(Q_t)$, which are determined by the linear PDE and geometry of N, L and S_t, are well-controlled as $t \to 0$.

The statement of the theorem is then that there exists t sufficiently small and ϵ_t so that the contraction mapping argument works.

This is a delicate balancing act since as $t \to 0$ parts of the manifold are collapsing, so the constants $C(\Delta_t), C(Q_t)$ above (which depend on t) can and typically do blow-up as $t \to 0$. To control this, we need to understand the Laplacian on N, L and S_t and introduce "weighted" Banach spaces so that tL gets rescaled to constant size (independent of t), and S_t resembles the union of two manifolds with a cylindrical neck (as we described earlier). It is also crucial to understand the relationship between the kernels and cokernels of the Laplacian on the *non-compact* N (without the intersection points), L and compact S_t: here is where connectedness is important so that the kernel and cokernel of the Laplacian is 1-dimensional. □

Remark In more challenging gluing problems it is not possible to show that the relevant map is a contraction, but rather one can instead appeal to an alternative fixed point theorem (e.g. Schauder fixed point theorem) to show that it still has a fixed point.

Example. Let N be a compact special Lagrangian 3-fold with a conical singularity modelled on the Harvey–Lawson T^2 cone. There is a topological condition that must be satisfied to ensure that a smoothing of the cone can be glued to N. This is purely an obstruction to having a Lagrangian gluing. Once this condition is satisfied you can perform the gluing a family of smooth compact special Lagrangians N_t such that $N_t \to N$ as $t \to 0$. It was then shown by Imagi that in fact all compact special Lagrangians near N must be found this way: this is sometimes called "surjectivity of the gluing". The reason why it works is because the T^2 cone is very simple in that its moduli space (as a cone) is as simple as it could be. Imagi's proof goes via the use of integral currents or varifolds.

Lecture 4: Yang–Mills flow and stability

5 Yang–Mills flow and stability

The study of special Lagrangians and Lagrangian mean curvature flow is motivated by ideas from *Mirror Symmetry.* I will not be precise about Mirror Symmetry (particularly since it is mainly conjectural anyway) but some features of the picture are the following. Given any compact Calabi–Yau n-fold M there is another compact Calabi–Yau n-fold W which is called the "mirror" of M. Since M and W are both Calabi–Yau they are both complex and symplectic. The feature of the mirror is that there is a correspondence between the symplectic geometry of M and the complex geometry of W, and vice versa. As we have seen, Lagrangian submanifolds are important and natural objects in the symplectic geometry of M, so we can ask what they correspond to on W. The answer is holomorphic vector bundles on W. Now we have seen special Lagrangian submanifolds on M as important examples of Lagrangian submanifolds, so again we can ask: what do they correspond to on W? The answer is holomorphic vector bundles which admit a Hermitian Yang–Mills connection (or equivalently, a Hermitian–Einstein metric). Now the question of when Hermitian Yang–Mills connections exist is fully understood, unlike the analogous situation in special Lagrangian geometry. In particular, the existence can be understood in terms of geometric flow, specifically *Yang–Mills flow*, and in terms of so-called *stability.* This stability is formalised in terms of the so-called Hitchin–Kobayashi correspondence, originally a conjecture and now a theorem often called the Donaldson–Uhlenbeck–Yau theorem.

Theorem 5.1. *A holomorphic vector bundle on W admits an (irreducible) Hermitian Yang–Mills connection (equivalently a Hermitian–Einstein metric), and thus the Yang–Mills flow exists for all time and converges, if and only if it is stable.*

Our aim now is to explain the meaning of this theorem and how the proof works, in particular the role that the geometric flow plays. As well as being interesting in its own right, it might (or perhaps should depending on how seriously you take Mirror Symmetry) provide some insight into the correponding existence and flow questions for special Lagrangians. In particular, it is these considerations which motivate the conjectures of Thomas–Yau and Joyce on Lagrangian mean curvature flow.

5.1 Yang–Mills connections

To start, let us consider a compact Riemannian manifold M with a vector bundle E on it, and suppose that A is a connection on E, which we view as a 1-form A on E with values in the endomorphisms of E. This defines a derivative $\mathrm{d}_A = \mathrm{d} + A$, which is an equivalent way to define the connection. The curvature of the connection $F_A = \mathrm{d}_A A = \mathrm{d}A + A \wedge A$ is then a 2-form with values in $\mathrm{End}(E)$. We can then define the following.

Definition 5.2. The Yang–Mills functional on connections on E is given by

$$YM(A) = \int_M |F_A|^2 \,\mathrm{vol}_M \,.$$

Clearly, $YM(A) \geq 0$ and $YM(A) = 0$ if and only if A is a flat connection, but what critical points does this functional have? A connection A is called a *Yang–Mills* connection if

$$\frac{\mathrm{d}}{\mathrm{d}t} YM(A + ta)|_{t=0} = 0$$

for all 1-forms a with values in $\mathrm{End}(E)$.

Again, we have only defined critical points, whereas we want to understand minimizers. Obviously flat connections are minimizers, but are there any others? That will motivate us later on.

It is straightforward to compute (which you should do) that

$$\frac{\mathrm{d}}{\mathrm{d}t} YM(A + ta)|_{t=0} = 2 \int_M \langle a, \mathrm{d}_A^* F_A \rangle \,\mathrm{vol}_M \,.$$

We therefore have the following result.

Lemma 5.3. *A connection A is Yang–Mills if and only if*

$$\mathrm{d}_A^* F_A = 0,$$

which is called the Yang–Mills equation.

Remark This equation comes from physics, and is related to the Standard Model in particle physics.

Notice that the Yang–Mills equation is a second order equation since F_A is first order in the connection. Recall that F_A also satisfies the Bianchi identity $\mathrm{d}_A F_A = 0$. Hence, F_A satisfies the elliptic first order system $\mathrm{d}_A F_A = \mathrm{d}_A^* F_A = 0$. It is not elliptic in A because the equation is invariant under natural transformations, called *gauge transformations*: at a point in M this is simply the action by an endomorphism of the fibre of E. This is an infinite dimensional group of transformations, but there is a way to apply a so-called *gauge fixing*, which ensures that the redundancy from gauge transformations is removed. Once this is achieved, we then have that the Yang–Mills equation (modulo gauge) is a second order elliptic system for the connection. This should now remind you of minimal submanifolds and the zero mean curvature equation.

Now that we have the Yang–Mills equation, we have a natural flow we can study.

Definition 5.4. We say a family A_t of connections on E satisfies Yang–Mills flow if

$$\frac{\partial}{\partial t} A = -\mathrm{d}_A^* F_A.$$

We see that Yang–Mills flow is the negative gradient flow of the Yang–Mills functional, and hence that critical points of the flow are precisely the Yang–Mills connections.

Again, the Yang–Mills flow as it stands is a second order equation which is not parabolic, because of gauge transformations. However, one can again see that modulo these gauge transformations the equation is parabolic for A. This should remind us of mean curvature flow.

It is worth computing how the curvature evolves along Yang–Mills flow:

$$\begin{aligned}\frac{\partial}{\partial t} F_A &= \frac{\partial}{\partial t}(\mathrm{d}A + A \wedge A)\\ &= \mathrm{d}(\frac{\partial}{\partial t}A) + \frac{\partial}{\partial t}A \wedge A + A \wedge \frac{\partial}{\partial t}A\\ &= -\mathrm{d}\mathrm{d}_A^* F_A - \mathrm{d}_A^* F_A \wedge A - A \wedge \mathrm{d}_A^* F_A\\ &= -\mathrm{d}_A \mathrm{d}_A^* F_A.\end{aligned}$$

Now, since F_A satisfies the Bianchi identity $\mathrm{d}_A F_A = 0$ (as $F_A = \mathrm{d}_A A$), we can rewrite this as follows.

Lemma 5.5. *Along Yang–Mills flow,*

$$\frac{\partial}{\partial t} F_A = -\mathrm{d}_A \mathrm{d}_A^* F_A = -(\mathrm{d}_A \mathrm{d}_A^* + \mathrm{d}_A^* \mathrm{d}_A) F_A = -\Delta_A F_A.$$

Hence, F_A satisfies a nonlinear heat (i.e. parabolic) equation.

This is a parabolic equation regardless of gauge transformations, and so is a useful tool for studying the Yang–Mills flow. The idea then is to start from a given connection and try to run the flow to see if we can find a Yang–Mills connection. In general this does not currently work that well, but in special cases it does as we shall now see.

5.2 Hermitian–Yang–Mills connections

We now specialise to the setting of a Calabi–Yau n-manifold M with Kähler form ω and holomorphic volume form Υ (though much of what I say works for Kähler manifolds). We consider complex vector bundles E endowed with a Hermitian metric h on the fibres and connections A which are compatible with h, so-called Hermitian connections, i.e.

$$d(h(u, v)) = h(\mathrm{d}_A u, v) + h(u, \mathrm{d}_A v)$$

for all sections u, v of E. We then impose the condition that

$$F_A \wedge \Upsilon = 0.$$

(This is equivalent to saying that F_A is of type $(1,1)$.) The reason is the following, recalling that E is holomorphic (namely that it admits a choice of complex structure so that $\pi : E \to M$ is holomorphic) if and only if it admits a Cauchy–Riemann operator $\bar\partial$ (often called a holomorphic structure).

Proposition 5.6. *If A is a Hermitian connection on E satisfying $F_A \wedge \Upsilon = 0$, then there is a unique holomorphic structure $\bar\partial$ on E such that the $(0,1)$ component of d_A is $\bar\partial$.*

Conversely, if E admits a holomorphic structure $\bar\partial$ then there is a unique Hermitian connection on E so that $\bar\partial = \mathrm{d}_A^{(0,1)}$ and $F_A \wedge \Upsilon = 0$. (This connection is known as the Chern connection).

This shows the one-to-one correspondence between certain connections on holomorphic vector bundles and Hermitian metrics on holomorphic vector bundles. This will be extremely important in what follows.

Our aim now is to understand what the Yang–Mills condition looks like for the connection A. So, given a Hermitian connection satisfying $F_A \wedge \Upsilon = 0$, it is easy to compute (and you should do it) that

$$F_A \wedge \frac{\omega^{n-2}}{(n-2)!} = (F_A \cdot \omega) * \omega - *F_A.$$

The point is that F_A can be decomposed into orthogonal components, so that the first part is a multiple of ω and the second part is orthogonal to ω (sort of, trace and trace-free parts):

$$F_A = \frac{(F_A \cdot \omega)}{n}\omega + \left(F_A - \frac{(F_A \cdot \omega)}{n}\omega\right).$$

So, let us now compute the condition for A to be Yang–Mills. Recall that we have that Bianchi identity $\mathrm{d}_A F_A = 0$ and $\mathrm{d}\omega = \mathrm{d} * \omega = 0$, so differentiating gives:

$$\begin{aligned}
0 &= \mathrm{d}_A F_A \wedge \frac{\omega^{n-2}}{(n-2)!} + F_A \wedge \mathrm{d}(\frac{\omega^{n-2}}{(n-2)!}) \\
&= \mathrm{d}_A\left(F_A \wedge \frac{\omega^{n-2}}{(n-2)!}\right) \\
&= \mathrm{d}_A((F_A \cdot \omega) * \omega - *F_A) \\
&= \mathrm{d}_A(F_A \cdot \omega) * \omega + (F_A \cdot \omega)\mathrm{d} * \omega - \mathrm{d}_A * F_A \\
&= \mathrm{d}_A(F_A \cdot \omega) * \omega - \mathrm{d}_A * F_A.
\end{aligned}$$

We therefore see that A is Yang–Mills, so $\mathrm{d}_A * F_A = 0$, if and only if $\mathrm{d}_A(F_A \cdot \omega) = 0$ since $*\omega \neq 0$, which is if and only if

$$F_A \cdot \omega = \lambda \,\mathrm{id}_E$$

constant (times the identity). This gives us the following definition/lemma.

Definition 5.7. A Hermitian connection A on (E, h) is Hermitian–Yang–Mills if and only if

$$F_A \wedge \Upsilon = 0 \quad \text{and} \quad F_A \cdot \omega = \lambda \,\mathrm{id}_E$$

for a constant λ.

Notice that this equation is now *first order* in A, rather than second order, and again can be viewed as an elliptic system (modulo gauge). So, we can ask whether this is anything like calibrated geometry, i.e. whether these connections are minimizers for the Yang–Mills functional. To see this we observe first that the constant λ is not arbitrary: in fact, it is determined by E.

By Chern–Weil theory, we know that $\frac{1}{2\pi i} \operatorname{tr} F_A$ represents the first Chern class of E, $c_1(E) \in H^2(M)$. Thus,

$$\frac{1}{2\pi i}\int_M \operatorname{tr}(F_A \cdot \omega) = \int_M c_1(E) \wedge *\omega := \deg(E),$$

the degree of E, which only depends on $c_1(E)$ and the cohomology class $[\omega]$ of ω. On the other hand,

$$\int_M \operatorname{tr}(\lambda \,\mathrm{id}_E) = \lambda \operatorname{vol}(M) \operatorname{rank}(E).$$

Therefore, since the volume of M is fixed, we see that the constant λ is proportional to (and completely determined by)

$$\mu(E) := \frac{\deg(E)}{\operatorname{rank}(E)},$$

which is topological.

Similar Chern–Weil theory considerations and the orthogonal decomposition for F_A we had before show that Hermitian–Yang–Mills connections are in fact minimizers for the Yang–Mills functional, so the parallels with calibrated geometry are becoming more apparent.

The idea then is to use the flow to find Hermitian–Yang–Mills connections. If we want to use a flow to study Hermitian–Yang–Mills connections, then we had better show that the flow preserves the condition $F_A \wedge \Upsilon = 0$:

$$\begin{aligned}\frac{\partial}{\partial t}(F_A \wedge \Upsilon) &= \frac{\partial}{\partial t}F_A \wedge \Upsilon \\ &= -\Delta_A F_A \wedge \Upsilon \\ &= -\Delta_A(F_A \wedge \Upsilon),\end{aligned}$$

where in the last part we used a special property of Calabi–Yau manifolds that Δ_A commutes with wedging with Υ. From this formula one can quickly deduce that the condition $F_A \wedge \Upsilon = 0$ is indeed preserved along the flow (by the maximum principle applied to the evolution of $|F_A \wedge \Upsilon|^2$ for example).

We are therefore in business to use Yang–Mills flow to study our problem.

5.3 Stability

Now we want to see what this stability condition is and what it has to do with the flow. The quantity $\mu(E)$, known as the slope, is in fact precisely what is needed to understand stability.

Definition 5.8. A holomorphic bundle (E, h) is stable if and only if for all proper coherent subsheafs E' of E we have

$$\mu(E') < \mu(E).$$

(A coherent sheaf is a weaker notion than holomorphic vector bundle for which things like first Chern class and rank, and thus μ, make sense.) We say that (E, h) is semistable if we replace the strict inequality by $\leq$.

Informally, a bundle is stable if it cannot be broken up into smaller pieces for which μ is larger. This is good because along the flow it would mean that the bundle does not break up into smaller subbundles (or subsheafs).

Remark Where does this condition come from? Unfortunately this will take me too long to explain, but it is from geometric invariant theory (GIT). The idea is that we can complexify the action of the gauge transformations and look at the orbits of this complexified group action. We can view the Hermitian–Yang–Mills connections as critical points for the norm squared of the moment map for the gauge group action, so we are motivated by the classical Kempf–Ness theorem: that the norm will have a minimum on the complexified orbit if and only if the orbit is closed. This is where stability comes from: it enables us to determine when the orbit will be closed (and this also shows the links to coherent sheaves, because we need to look at "limits" of holomorphic bundles). Thus the Hitchin–Kobayashi correspondence can be seen as an infinite-dimensional analogue of the Kempf–Ness theorem.

We now know what stability is and so we understand the statement of the existence theorem from the start of the section. The point now is that this stability condition could (or perhaps should dependly on how seriously you take Mirror Symmetry) provide motivation for a condition for which Lagrangian mean curvature flow exists for all time and converges to a special Lagrangian. This is precisely where the Thomas–Yau conjecture comes from.

5.4 Long time behaviour of the flow

Now that we have the ingredients, how does the proof go? Well, there are different ways to do it, but let me explain the flow approach, which is due to Donaldson.

The first idea is to recast the problem in terms of Hermitian metrics on E, recalling that Hermitian metrics on holomorphic vector bundles are in one-to-one correspondence with the Hermitian connections we want to study. A Hermitian metric on E at each point of M is just a positive definite Hermitian matrix. If we look at the space of $m \times m$ Hermitian matrices $\mathrm{Herm}^+(m)$, then this can be viewed as

$$\mathrm{Herm}^+(m) \cong \mathrm{GL}(m, \mathbb{C}) / \mathrm{U}(m),$$

which is a symmetric space of non-compact type, and naturally has a *negatively curved* metric. Hence, we can view the space of Hermitian metrics on E, $\mathrm{Herm}^+(E)$, as a negatively curved space.

This means that Yang–Mills flow can be recast as a parabolic flow in a negatively curved space, and this is a good place to apply the maximum principle, since everything should contract. In fact, Donaldson introduced a functional which is monotone along Yang–Mills flow and gives good control on the evolving metrics. (Of course, the Yang–Mills functional is monotone along the flow, but gives only L^2 control on the curvature of the connection, which is quite weak.) Applying the maximum principle then quickly gives long time existence of the flow (regardless of stability or not!).

So what about stability? The point is that stability gives an a priori bound on the functional and thus on the size of the metrics. This ensures that not only do we get long-time existence, but convergence to a critical point (a Hermitian–Einstein metric), which is equivalent to a Hermitian–Yang–Mills connection. Conversely, it is straightforward to see that if there is a Hermitian–Yang–Mills connection then the Yang–Mills flow will converge to it (since we have long-time existence) and then the fact that the bundle is stable is something one can prove using algebraic geometry (and is, in some sense, the "easy" part of the Hitchin–Kobayashi correspondence).

Remark One may ask what happens along the flow if E is not stable. We still have long time existence, but the flow will not converge. Instead it will break up the bundle into smaller (semi)stable pieces: this is called the Harder–Narasimhan filtration.

Lecture 5: Lagrangian mean curvature flow

6 Lagrangian mean curvature flow

Lagrangian mean curvature flow is potentially a powerful tool for answering major open problems in the study of Calabi–Yau manifolds in particular, but also more generally in symplectic topology. The clearest problem that it could tackle is the questions of existence and uniqueness of special (or simply minimal) Lagrangians. This would have possibly have implications for the construction of new invariants of Calabi–Yau manifolds from "counting" special Lagrangians, the existence of special Lagrangian fibrations as suggested by the SYZ conjecture, and the nearby Lagrangian conjecture (that any closed exact Lagrangian in a cotangent bundle is Hamiltonian isotopic to the zero section).

With the discussion of Hermitian–Yang–Mills connections in hand, and motivated by Mirror Symmetry, we now have a possible route to solving these problems by seeing if something similar can happen for special Lagrangians via a flow approach and a notion of stability. We therefore want to understand some of the basics of Lagrangian mean curvature flow and several key results in the field. I will also discuss a few open questions.

6.1 Existence of the flow

Of course, mirror to the Yang–Mills flow, the first thing to ask is: is there such a thing as Lagrangian mean curvature flow? In other words, does mean curvature flow preserve the Lagrangian condition? The answer, in general, is of course no, since the Lagrangian condition is a symplectic topology statement, and mean curvature flow is a Riemannian geometry object and so they have nothing to do with each other most of the time. However, there are settings where it works.

Example. Let γ be a curve in $\mathbb{C} \cong \mathbb{R}^2$. Then γ is automatically Lagrangian since the Kähler form is a 2-form and so must vanish on γ for dimension reasons. Recall the curve shortening flow we saw at the beginning of the course, given by the variation

$$\frac{\partial F}{\partial t} = \kappa$$

where κ is the curvature of the evolving curve. This is the mean curvature flow for plane curves and so trivially preserves the Lagrangian condition, so this is the simplest example of Lagrangian mean curvature flow. (Even though it is simple, I think it can be instructive and so I encourage you to learn more about curve shortening flow.)

Of course, the situation of curves in the plane (or in a surface) is restrictive, so we want a more general setting, which is provided by the following result due to Smoczyk.

Theorem 6.1. *If M is Calabi–Yau (in particular if M is $\mathbb{C}^n$) and N is a compact Lagrangian then the solution N_t to mean curvature flow is Lagrangian for all t.*

Remark This result also holds if we just assume that M is Kähler–Einstein. Together with Pacini, I showed that this result can be extended in a certain sense to any symplectic manifold with a compatible almost complex structure.

I will just sketch the proof as it is a long calculation, but let me give the idea why you should believe it. Remember our magic formula:

$$H = J\nabla\theta.$$

We then want to see if we let $i_t = F(.,t)$ (so that $i_t(N) = N_t$) then

$$\frac{\partial}{\partial t} i_t^* \omega = 0.$$

Since it is zero initially and therefore would then be zero for all time. Using our formula and Cartan's formula we see that

$$\begin{aligned}\frac{\partial}{\partial t} i_t^*\omega &= i_t^*\mathcal{L}_H\omega \\ &= i_t^*\mathrm{d}(H \lrcorner \omega) \\ &= i_t^*\mathrm{d}(J\nabla\theta \lrcorner \omega) \\ &= -i_t^*\mathrm{d}(\mathrm{d}\theta) \\ &= 0.\end{aligned}$$

However, this argument is bogus! The point is that I cannot use $H = J\nabla\theta$ since that needs the Lagrangian condition, and so I am assuming what I need to prove to deduce the conclusion! However, it is still true at $t = 0$ and so infinitesimally the flow preserves the Lagrangian condition. If this were just an ODE problem that would be enough, but it is a PDE problem and so we cannot just conclude that if it starts zero with zero velocity then it stays zero.

The real argument is to let $f = |i_t^*\omega|^2 \geq 0$. Then you show that this satisfies

$$\frac{\partial}{\partial t} f \leq -\Delta f + Cf$$

for a constant $C \geq 0$. Applying the maximum principle to this parabolic inequality, we know that the maximum of f is decreasing in time, but it is zero initially so it must stay zero.

Example. As an example, we saw that the Clifford torus in $\mathbb{C}^2 \cong \mathbb{R}^4$ is Lagrangian and simply shrinks under mean curvature flow, which means that it stays Lagrangian under the flow, as we know by the theorem above.

Now that we know that mean curvature flow preserves the Lagrangian condition in Calabi–Yaus, let us restrict to that setting. Here, the flow for $F : N \times (0, \epsilon) \to M$, with N initially Lagrangian, becomes

$$\frac{\partial F}{\partial t} = H = J\nabla\theta.$$

Facts from symplectic geometry (this is a symplectic isotopy) then show that the Maslov class $[\mathrm{d}\theta]$ of N is preserved along the flow. Recall that special Lagrangians are automatically zero Maslov, since their Lagrangian angle is constant and therefore trivially a single valued function. Hence, that we must restrict to initial conditions which are zero Maslov for the flow to have any chance of converging.

If N is zero Maslov, then θ is a well-defined function for all time and so Lagrangian mean curvature flow becomes a *Hamiltonian isotopy*. This is great, because it means we can study the question: when does a special Lagrangian exist in a given Hamiltonian isotopy class?

6.2 Fundamentals

For the Yang–Mills flow we said that understanding how the curvature evolves is important, so here in Lagrangian mean curvature flow it would be good to understand how H evolves. However, we know that $H = J\nabla\theta$ in the zero Maslov class case, and it is perhaps even better to study how θ evolves as this will be a scalar PDE which is much easier to study than a PDE system.

Proposition 6.2. *If N is zero Maslov then along the Lagrangian mean curvature flow the Lagrangian angle satisfies*

$$\frac{\partial}{\partial t}\theta = -\mathrm{d}^*\mathrm{d}\theta = -\Delta\theta.$$

Thus, at the level of the angle, we have a (nonlinear) heat equation. This will clearly be very important.

Proof. Recall that

$$\Upsilon|_N = e^{i\theta}\,\mathrm{vol}_N,$$

so in general we have

$$i_t^*\Upsilon = e^{i\theta_t} i_t^*\,\mathrm{vol}_{N_t}$$

where θ_t is the Lagrangian angle of $N_t = i_t(N) = F(N,t)$.

On the one hand, we can compute using Cartan's formula, $\mathrm{d}\Upsilon = 0$, the fact that Υ is an $(n,0)$-form and $H = J\nabla\theta$,

$$\begin{aligned}\frac{\partial}{\partial t} i_t^*\Upsilon &= i_t^*\mathcal{L}_H\Upsilon \\ &= i_t^*(\mathrm{d}(H\lrcorner\Upsilon) + H\lrcorner\mathrm{d}\Upsilon) \\ &= \mathrm{d}(i_t^*(J\nabla\theta_t\lrcorner\Upsilon)) \\ &= i\mathrm{d}(i_t^*(\nabla\theta_t\lrcorner\Upsilon)).\end{aligned}$$

Now, $\nabla\theta_t$ is tangential and

$$\nabla\theta_t\lrcorner i_t^*\,\mathrm{vol}_{N_t} = *\mathrm{d}\theta_t,$$

where $*$ is the Hodge star induced from the metric on N_t. Therefore,

$$\begin{aligned}\frac{\partial}{\partial t} i_t^*\Upsilon &= i\mathrm{d}(e^{i\theta_t} * \mathrm{d}\theta_t) \\ &= e^{i\theta_t}(-\mathrm{d}\theta_t \wedge *\mathrm{d}\theta_t + i\mathrm{d} * \mathrm{d}\theta_t).\end{aligned}$$

On the other hand, we have by the first variation formula,

$$\frac{\partial}{\partial t}(e^{i\theta_t} i_t^*\,\mathrm{vol}_{N_t}) = e^{i\theta_t}\left(-|H|^2 i_t^*\,\mathrm{vol}_{N_t} + i\frac{\partial\theta_t}{\partial t} i_t^*\,\mathrm{vol}_{N_t}\right).$$

Hence, we have computed the derivative of the same thing two ways, and if we divide both answers by $e^{i\theta_t}$ we can compare imaginary parts on both sides to deduce that

$$\frac{\partial\theta_t}{\partial t} i_t^*\,\mathrm{vol}_{N_t} = \mathrm{d} * \mathrm{d}\theta_t.$$

Applying the Hodge star gives the result we wanted since $*\mathrm{d}* = -\mathrm{d}^*$ on 1-forms. (Notice that comparing real parts just tells us that $|H|^2 = |\mathrm{d}\theta_t|^2$, which we knew since $H = J\nabla\theta$.) □

I cannot emphasise how important this result is. It really provides the foundation for much of the study in Lagrangian mean curvature flow. In particular, it motivates the study of the following Lagrangians.

Definition 6.3. A compact oriented Lagrangian N in a Calabi–Yau manifold is called *almost calibrated* if $\operatorname{Re}\Upsilon|_N > 0$, where Υ is the holomorphic volume form. Equivalently, it says that N is zero Maslov and the Lagrangian angle satisfies $\cos\theta > 0$, so we can assume that θ lies in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$.

Since N is compact, this condition forces $\cos\theta \geq \epsilon > 0$ for some $\epsilon > 0$ so we can also use this condition to mean almost calibrated as it applies to non-compact Lagrangians.

Example. The grim reaper curve

$$\gamma = \{(x, -\log\cos x) \in \mathbb{R}^2 \,:\, x \in (-\frac{\pi}{2}, \frac{\pi}{2})\}$$

has Lagrangian angle taking all values in the range $(-\frac{\pi}{2}, \frac{\pi}{2})$, so this is *not* almost calibrated.

Lemma 6.4. *If N is almost calibrated, then along Lagrangian mean curvature flow N_t is almost calibrated for all t.*

Proof. We know that $\dot{\theta} = -\Delta\theta$, which is a parabolic equation, so the maximum principle tells us the maximum of θ is decreasing in time and that the minimum of θ is increasing in time (by applying the maximum principle to $-\theta$). Hence, if θ takes values $(-\frac{\pi}{2}, \frac{\pi}{2})$ initially it must continue to take values in the interval. □

We will see the role of almost calibrated Lagrangians very shortly.

6.3 Singularities

Now we will turn to the study of singularities in Lagrangian mean curvature flow, since in any flow understanding the singularities is a key step to analysing the long-time behaviour of the flow.

We have already seen the simplest kind of flow which becomes singular in finite time: self-shrinkers. (You should imagine spheres or cylinders shrinking.) We will have seen that if N is a self-shrinker then (by rescaling the initial N appropriately) the solution is

$$N_t = \sqrt{1-t}N$$

which becomes singular at $t = 1$. Notice that there is a very special type of self-shrinker which would be a stationary solution (i.e. a minimal Lagrangian in our case) so $N_t = N$ for all t, but we would also need it to be dilation invariant, i.e. a cone. So, in terms of the flow, self-shrinkers generalise minimal cones.

It is more usual to have the singularity happen at time 0 and for the flow to exist for all negative times (a so-called *ancient solution*), so we set $s = t - 1$ for $t < 1$ and obtain

$$N_s = \sqrt{-s}N$$

for $s < 0$. Now, notice that if we choose any increasing sequence of positive numbers $\lambda_j \to \infty$ then if we define the *Type I blow-up* by

$$N^j_s = \lambda_j N_{\lambda_j^{-2}s}$$

then

$$N^j_s = \lambda_j\sqrt{-\lambda_j^{-2}s}N = \sqrt{-s}N = N_s$$

is independent of the λ_j.

These facts, together with Huisken's monotonicity formula motivates us to study the Type I blow-up for any solution N_t of mean curvature flow: explicitly the blow-up around time $t = T$ and point p given by a sequence $\lambda_j \to \infty$ is

$$N^j_s = \lambda_j(N_{T+\lambda_j^{-2}s} - p)$$

for all $s < 0$. (This looks very similar to way one defines the tangent cone of a submanifold, as it should.) The point is that this should converge to a self-shrinker as $j \to \infty$ and hence give us the "local model" (or "first approximation") for the behaviour of the flow near the singular point at the singular time.

The key theorem of Neves is the following, which says that in the zero Maslov class case we do not see all the self-shrinkers as Type I blow-ups, but just the minimal cones.

Theorem 6.5 (Neves). *Let N be a zero Maslov class Lagrangian and let N^j_s be a Type I blow-up of the Lagrangian mean curvature flow N_t starting at N. Then $N^j_s \to \cup_{k=1}^m C_k$ as $j \to \infty$, where C_k is a special Lagrangian cone with phase $e^{i\theta_k}$.*

This is the main known structure theorem for the singularities of the flow, which says the local model should be a union of minimal Lagrangian cones. Notice that the phases could be different, which means that the union of cones may well not be calibrated as you need a single constant phase to be calibrated.

A consequence of our discussion above is that the "leading order" behaviour at a singularity of the flow should be modelled on a (smooth) self-shrinker. However, we have the following result, which is a key (implicit) ingredient in the theorem of Neves above.

Proposition 6.6. *A smooth zero Maslov Lagrangian self-shrinker is a plane.*

This is easy to prove (using a maximum principle argument), but I will not do it for lack of time. This proposition means that the fastest kind of singularities, in the sense of the blow-up rate of the norm of the second fundamental form at the singularity (so-called *Type I singularities*), do not occur for zero Maslov Lagrangian mean curvature flow.

This is good news because it has been suggested (in other contexts) that Type I singularities are the generic type of singularity. Thus, one might hope that by perturbing the initial conditions there will be no finite time singularities in zero Maslov Lagrangian mean curvature flow.

On the other hand, it is also bad news as it means that singularities must form slowly (so called *Type II singularities*) and the model will not be "local". What I mean is, the reason why the singularity is forming and part of the Lagrangian is collapsing is not because it is locally modelled on a self-shrinker, but because global effects are causing that part of the Lagrangian to become singular. This global nature of the problem makes it much more difficult to analyse, and we currently do not have a way to tackle the issue of Type II singularities well in any geometric flow.

Example. A known model for a Type II singularity is a so-called translating solution to mean curvature flow, which moves simply by translations under the flow. The simplest example is the grim reaper curve introduced earlier. As we noted, this example is not almost calibrated, so one might hope we can rule out translating solutions as singularity models using the almost calibrated condition. However, we would be mistaken because Joyce–Lee–Tsui constructed Lagrangian translating solutions in $\mathbb{C}^n$ for all $n \geq 2$ which are almost calibrated (in fact, the oscillation of the Lagrangian angle can be made arbitrarily small).

6.4 Conjectures

The first main conjecture in Lagrangian mean curvature flow comes directly from the Yang–Mills flow via Mirror Symmetry. The proposal, due to Richard Thomas, is (roughly speaking) the following. He wants to obtain a gauge group action, just like in the Yang–Mills setting, so considers pairs (N, A) where N is Lagrangian and A is a flat connection on a U(1)-bundle on N. One naturally obtains a U(1) action on these pairs (since one can act on the flat connection). Formally complexifying this U(1) action, as one wants to do in GIT, leads to Hamiltonian deformations of N, as we would like. Thomas also shows that $\operatorname{Im}\Upsilon|_N$ is the moment map for this action, so that special Lagrangians correspond precisely to zeros of the moment map.

Now that one has the GIT set up one would like to mimic the Hitchin–Kobayashi correspondence from Yang–Mills theory on Calabi–Yaus, and so one has the following conjecture, due to Thomas–Yau.

Conjecture 6.7 (Thomas–Yau)**.** *Let N be a compact, embedded, almost calibrated Lagrangian in a compact Calabi–Yau manifold M. There exists a unique special Lagrangian in the Hamiltonian isotopy class of N, and thus the Lagrangian mean curvature flow exists for all time and converges starting at N, if and only if N is stable.*

It is important to note that Thomas–Yau realised when they stated this that the conjecture would not be true if N is *not* almost calibrated. This is borne out by the following breakthrough result of Neves.

Theorem 6.8 (Neves)**.** *Let N be a compact embedded Lagrangian in a Calabi–Yau 2-fold M. There exists a compact embedded Lagrangian N' Hamiltonian isotopic to N so that the Lagrangian mean curvature flow starting at N' develops a finite-time singularity.*

Remark Although this theorem is stated in 2 dimensions, it clearly extends to all Calabi–Yau n-folds for $n \geq 2$. It is, however, false in dimension 1, since this is curve shortening flow and it is known that the flow starting at any compact embedded curve on T^2 homologous to a geodesic loop will exist for all time and converge (to a geodesic).

Remark The Lagrangian N' is constructed explicitly in the theorem as a perturbation of N and is *not* almost calibrated even if N is, so this theorem does *not* disprove the Thomas–Yau conjecture.

Even though this theorem does not disprove the Thomas–Yau conjecture, Joyce (and many others) believe the conjecture to be false. Joyce, however, has suggested that the conjecture is "morally" true, in that there should a notion of stability which ensures the long-time existence of a Lagrangian mean curvature flow, but crucially allowing *surgeries*. I will briefly discuss the sort of thing that Joyce has in mind at the end.

Of course, to make sense of what the Thomas–Yau conjecture says, I have to tell you what the stability condition is.

Definition 6.9. For a compact almost calibrated Lagrangian N we define $\phi(N) \in (-\frac{\pi}{2}, \frac{\pi}{2})$ by

$$e^{i\phi(N)} = \frac{\int_N \Upsilon}{|\int_N \Upsilon|},$$

that is, $\phi(N)$ is the argument of the complex number $\int_N \Upsilon$. It must lie in the interval stated because the real part of $\int_N \Upsilon$ is strictly positive.

Notice that $\phi(N)$ only depends on the homology class of N since Υ is closed and that

$$\int_N e^{-i\phi(N)}\Upsilon = \left|\int_N \Upsilon\right| \in (0, \mathrm{vol}(N)]$$

and is equal to the volume of N if and only if N is special Lagrangian with phase $e^{i\phi(N)}$.

In particular, if N is Hamiltonian isotopic to a special Lagrangian then $\phi(N) = 0$, so we need only consider almost calibrated Lagrangians with $\phi(N) = 0$ in studying almost calibrated Lagrangian mean curvature flow.

We can now define the notion of stability given in Thomas–Yau.

Definition 6.10. A compact, embedded, almost calibrated Lagrangian N in a Calabi–Yau manifold with $\phi(N) = 0$ is stable if for all N_-, N_+ compact almost calibrated Lagrangians so that N is Hamiltonian isotopic to the (graded Lagrangian) connect sum $N_-\#N_+$ we have

- $[\phi(N_-), \phi(N_+)] \not\subseteq (\inf_N \theta, \sup_N \theta)$ or
- $\mathrm{vol}(N) \leq \displaystyle\int_{N_-} e^{-i\phi(N_-)}\Upsilon + \int_{N_+} e^{-i\phi(N_+)}\Upsilon \leq \mathrm{vol}(N_-) + \mathrm{vol}(N_+)$.

The maximum principle ensures that if the first condition is satisfied by N then it is satisfied by N_t for all t. Mean curvature flow decreases volume, so if N satisfies the second condition then so does N_t for all t.

The second condition is easier to understand: it says in particular that if N breaks up into two almost calibrated pieces then the union of those pieces must have larger volume than N. Since the flow is volume decreasing this should prevent N breaking up into the union of N_- and N_+. The first condition always ensures that N does not break up as at least one of $\phi(N_\pm)$ cannot lie in the limiting interval determined by the angle θ.

I should say that actually rather little is known about *either* direction of the Thomas–Yau conjecture. As I said in the Yang–Mills case, there was the "easy" direction of showing that if there was a Hermitian–Yang–Mills connection then the bundle was stable. We do not know this is the case. There are some partial results (due to Thomas–Yau) that the existence of a special Lagrangian in a Hamiltonian isotopy class implies some form of stability, but only under similar restrictions to the Thomas–Yau uniqueness result (e.g. for compact Lagrangians N which are spin and have $b^2(N) = 0$).

For the other direction, again there are some partial results of Thomas–Yau supporting the conjecture, but these are essentially 1-dimensional situations (either by thinking about curves or by using symmetries to reduce to a situation involving evolving curves). By this I mean, the results show both that stability implies long-time existence and convergence and that instability implies the formation of finite time singularities.

Remark Some concrete evidence that singularities can still form in the stable almost calibrated case is given by the existence of the almost calibrated translating solutions of Joyce–Lee–Tsui.

Overall, there is not a lot of evidence for the Thomas–Yau conjecture, and as I said it is thought by many to be false. However, Joyce has an updated version of the conjecture which can be very roughly stated as below.

Conjecture 6.11. *Let N be a compact Lagrangian which is spin, satisfies $b^2(N) = 0$ and is generic in its Hamiltonian isotopy class in a compact Calabi–Yau manifold M. There exists a unique special Lagrangian isomorphic to N, and the Lagrangian mean curvature flow, with a finite number of certain types of singularities and surgeries, exists for all time and converges starting at N, if and only if N is stable.*

On the one hand it is much more ambitious, as it does not restrict to studying almost calibrated Lagrangians, but on the other hand it is much more negative in that it says one must consider a flow with surgeries.

Remark The special Lagrangian limit here may not be smooth and it may not lie in the same Hamiltonian isotopy class as N. It only has to be isomorphic to N as an object in the derived Fukaya category.

To explain this conjecture I need to explain roughly what the flow with surgeries means and what the stability condition is.

I can say straight away that I cannot tell you what the stability condition is because it is part of the conjecture! This does not quite make the conjecture vacuous or a tautology because Joyce suggests a fairly precise form for the stability condition, namely it should be a so-called *Bridgeland stability condition* on the derived Fukaya category.

The singularities and surgeries involved are of four types:

- gluing in an exact (and thus zero Maslov) Lagrangian self-expander at an immersed point;
- "neck pinches" given by Lawlor necks;
- formation/smoothing of particularly simple isolated conical singularities (known as *stable* conical singularities – the Harvey–Lawson T^2 cone we saw earlier is an example);
- collapsing of pieces of the Lagrangian which are isomorphic to 0 (for example pieces given by the *Whitney sphere*, which is an immersed Lagrangian S^2 in $\mathbb{C}^2$ with a single immersed double point).

The last type of singularity (which is the sort occurring in the theorem of Neves constructing finite time singularities) can be ruled out by the almost calibrated condition , but the others are conjectured (by Joyce) to still be possible under Lagrangian mean curvature flow, even with the almost calibrated condition.

So the challenge now is to try to prove Joyce's conjecture. This will definitely be very difficult, and involve a collaborative effort between symplectic topologists and geometric analysts. It is an exciting topic to study and full of open questions worth exploring.