Stability, conifolds and G$_2$ geometry

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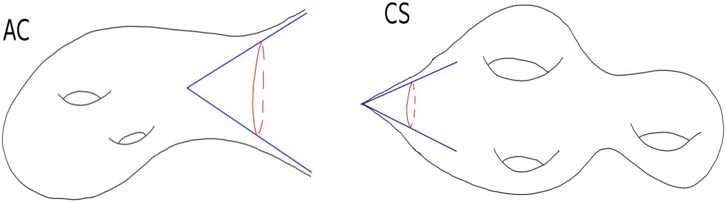
20 February 2013

Introduction

Conifolds



G_2 geometry

- conifolds $\leftrightarrow$ solutions to first-order nonlinear PDE $F(\alpha) = 0$
- linearise on cone $C \rightsquigarrow$ first-order linear PDE $G(\alpha_C) = 0$
- α_C homogeneous $\rightsquigarrow$ $H(\alpha_L) = \mu\alpha_L$ on link L

Stability index: certain count $\mathrm{ind}(C)$ of eigenvalues of H on L

Moral and motivation

Moral: $\operatorname{ind}(C)$ controls many aspects of conifolds in G_2 geometry

- Deformations
- Gluing
- Existence
- Uniqueness

Motivation

- Natural class of non-compact/singular manifolds
- Examples of AC conifolds
- Constructing examples is very difficult
- Models for how singularities develop
- M-theory – singularities are crucial

G_2 manifolds

(M^7, φ) with φ distinguished 3-form

- $\mathbb{R}^7$: $\varphi_0(u, v, w) = g_0(u \times v, w)$
- $G_2 = \mathrm{Stab}(\varphi_0)$
- M^7 oriented $\rightsquigarrow$ oriented isomorphism $\iota_p : T_pM \to \mathbb{R}^7$
- $\rightsquigarrow \Lambda^3_+ T^*_p M = \mathrm{GL}_+(7, \mathbb{R})$-orbit of $\iota_p^* \varphi_0$
- φ section of $\Lambda^3_+ T^* M \rightsquigarrow$ metric g_φ

Definition

(M^7, φ) *is a* G_2 *manifold if*

$$\mathrm{d}\varphi = \mathrm{d}^*_\varphi \varphi = 0 \;\Leftrightarrow\; \nabla_\varphi \varphi = 0 \;\Leftrightarrow\; \mathrm{Hol}(g_\varphi) \subseteq G_2$$

Linearised problem: essentially $\mathrm{d}\alpha = \mathrm{d}^*_\varphi \alpha = 0$ for 3-form α

Coassociative 4-folds

(M^7, φ) G_2 manifold

- $V \subseteq T_pM$ oriented 4-plane $\Rightarrow *_\varphi\varphi|_V \leq \mathrm{vol}_V$
- $\mathrm{d}*_\varphi\varphi = 0 \Rightarrow *_\varphi\varphi$ is a calibration

Definition

$N^4 \subseteq M^7$ *coassociative* $\Leftrightarrow *_\varphi\varphi|_N = \mathrm{vol}_N \Leftrightarrow \varphi|_N = 0$

- N volume-minimizing: $N' \in [N] \Rightarrow$

$$\mathrm{vol}(N') = \int_{N'} \mathrm{vol}_{N'} \geq \int_{N'} *_\varphi\varphi = \int_N *_\varphi\varphi = \mathrm{vol}(N).$$

- v normal vector field $\leftrightarrow v \lrcorner \varphi$ self-dual 2-form

Linearised problem: $\mathrm{d}\alpha = 0$ for self-dual 2-form α

Products and cones

Products

$M^7 = \mathcal{S}^1 \times Z^6$ G_2 manifold, $N^4 \subseteq \mathcal{S}^1 \times Z^6$ coassociative $\rightsquigarrow$

- (Z, J, ω, Ω) Calabi–Yau 3-fold
- $\varphi = d\theta \wedge \omega + \operatorname{Re}\Omega$ and $*_\varphi \varphi = \frac{1}{2}\omega \wedge \omega - d\theta \wedge \operatorname{Im}\Omega$
- $N^4 \subseteq Z^6 \Leftrightarrow N$ complex surface, i.e. $\frac{1}{2}\omega \wedge \omega|_N = \mathrm{vol}_N$
- $N^4 = \mathcal{S}^1 \times L^3 \Leftrightarrow L$ special Lagrangian, i.e. $\omega|_L = \operatorname{Re}\Omega|_L = 0$

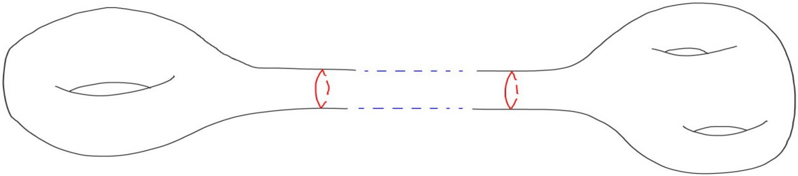
Cones

$N^4 = \mathbb{R}^+ \times L^3 \subseteq \mathbb{R}^+ \times Z^6 = M^7$

- M^7 G_2 cone $\Leftrightarrow (Z, J, \omega, \Omega)$ nearly Kähler
- N^4 coassociative cone $\Leftrightarrow L \subseteq Z$ Lagrangian, i.e. $\omega|_L = 0$
- $Z = \mathcal{S}^6$, $L \subseteq \mathcal{S}^5 \subseteq \mathcal{S}^6$ Hopf lift of holomorphic curve in $\mathbb{CP}^2$

Twisted connected sums

(Kovalev 2003, CHNP 2012) Take $Z_\pm$ asymptotically cylindrical CY 3-folds



- $Z^6_\pm \sim \mathcal{S}^1 \times \mathbb{R}^+ \times Y^4_\pm$, $Y^4_\pm$ K3 surface
- $\mathcal{S}^1 \times Z^6_\pm \sim \mathcal{S}^1 \times \mathcal{S}^1 \times \mathbb{R}^+ \times Y^4_\pm$
- "Twisted connected sum": swap circles plus hyperkähler rotation $\rightsquigarrow (M^7, \psi)$, $\mathrm{d}\psi = 0$, $\mathrm{d}^*_\psi \psi$ "small"
- (Joyce 1994) Perturb $\psi \rightsquigarrow (M, \varphi)$ holonomy G_2 manifold
- Smooth complex surfaces in $Z_\pm \rightsquigarrow$ coassociatives in (M, φ)

Conifolds

Conifolds: asymptotic cone C with link L

G$_2$ conifolds

- (Bryant–Salamon 1989) AC $\Lambda^2_+ T^*\mathcal{S}^4$, $\Lambda^2_+ T^*\mathbb{CP}^2$ with $L = \mathbb{CP}^3$, $\mathrm{SU}(3)/T^2$
- (Bryant–Salamon 1989) AC $\mathbb{S}(\mathcal{S}^3)$, $L = \mathcal{S}^3 \times \mathcal{S}^3$
- CS – no known examples

Coassociative conifolds

- (Harvey–Lawson 1982) AC $N \cong \mathbb{R}^2$-bundle over $\mathcal{S}^2$, L "squashed" $\mathcal{S}^3$
- (Harvey–Lawson 1982) AC $N \cong \mathbb{R} \times \mathcal{S}^3$, one end L "squashed" $\mathcal{S}^3$ and other end $\mathbb{R}^4$
- (L. 2006) AC from certain cones
- (L. 2012) First CS examples

Stability index

Coassociative cone $C^4 \subseteq \mathbb{R}^7$, Lagrangian link $L^3 \subseteq \mathcal{S}^6$

- $\mathrm{d}\alpha_C = 0$, α_C self-dual 2-form on C
- $\alpha_C = r^\lambda(r^2 {*}\alpha_L + r\mathrm{d}r \wedge \alpha_L)$, α_L 1-form on L
- $\rightsquigarrow {*}\mathrm{d}\alpha_L = (\lambda + 2)\alpha_L$, $\mathrm{d}{*}\alpha_L = 0$
- $m_L(\lambda)$ dimension of space of solutions α_L
- $m_L(-2) = b^1(L)$, $m_L(0)$ constant growth, $m_L(1)$ linear growth

Definition

Let $\mathcal{C}$ be the orbit of C under $\mathrm{G}_2 \ltimes \mathbb{R}^7$. The stability index

$$\mathrm{ind}(C) = \sum_{\lambda \in (-1,1]} m_L(\lambda) - \dim \mathcal{C} \geq 0$$

Generalise: replace $\mathcal{C}$ by deformation family of C

CS deformations

Theorem (L. 2007)

N CS coassociative in $(M, \varphi) \Rightarrow$

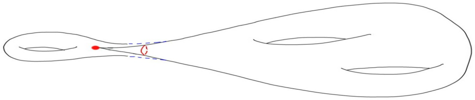
- *$\exists$ finite-dimensional $\mathcal{I}$, $\mathcal{O}$ with* $\dim \mathcal{O} \leq \operatorname{ind}(C)$
- *$\exists$ smooth map $\pi : \mathcal{I} \to \mathcal{O}$*

such that moduli space of deformations $\mathcal{M}(N) \cong \pi^{-1}(0)$ locally.

- $\operatorname{ind}(C) = 0 \Rightarrow \mathcal{M}(N)$ smooth
- $\operatorname{ind}(C) = 0 \Rightarrow N$ "stable" under deformations of φ, i.e. given φ_s there exists N_s with $\varphi_s|_{N_s} \equiv 0$
- $\operatorname{ind}(C)$ measures obstructions to deforming N
- Key idea: $\mathcal{M}(N) \cong F^{-1}(0)$ with $\operatorname{Coker}(\mathrm{d}F|_0) = \mathcal{O}$ and $\operatorname{ind}(C)$ part of $\operatorname{ind}(\mathrm{d}F|_0)$

Gluing

N_+ CS coassociative in (M,φ) and tN_- AC coassociative in $\mathbb{R}^7$



- v dilation vector field $\rightsquigarrow a = [v \lrcorner \varphi_0|_{N_-}] \in H^2(N_-)$
- Natural projections $\pi_\pm : H^2(N_\pm) \to H^2(L)$

Theorem (L. 2012)

$\pi_-(a) \in \operatorname{Im} \pi_+$, $\operatorname{ind}(C) = 0 \Rightarrow$ *can always glue to get* $X_t \cong X$ *and*

$$\dim \mathcal{M}(X) = \dim \mathcal{M}(N_+) + \dim \mathcal{M}(N_-)$$

- Expected codim of $\mathcal{M}(N_+)$ in $\mathcal{M}(X)$ is $\dim \mathcal{M}(N_-) + \operatorname{ind}(C)$
- Higher $\operatorname{ind}(C) \rightsquigarrow$ "less likely" N_+ arises as limit of X_t
- Key idea: need φ small on $N_+ \# tN_- \rightsquigarrow$ obstructions measured by $\operatorname{ind}(C)$

Existence

Theorem (L. 2012)

- *M twisted connected sum of $\mathcal{S}^1 \times Z_\pm$*
- *$Y \subseteq Z_\pm$ complex surface with ordinary double points*

$\Rightarrow \exists$ *CS coassociative deformation N of Y in M*

- $C \cong \{(0, z_1, z_2, z_3) \in \mathbb{C}^3 \ : \ a_1 z_1^2 + a_2 z_2^2 + a_3 z_3^2 = 0\}$
- $L \cong \mathbb{RP}^3$ Hopf lift of $\Sigma \cong \mathbb{CP}^1$ in $\mathbb{CP}^2$
- Fourier expansion of α_L such that $*\mathrm{d}\alpha_L = (\lambda + 2)\alpha_L$
- $\alpha_L \leftrightarrow$ eigenfunctions of Δ plus $H^0(K_\Sigma \otimes H^{\lambda+2})$
- $\mathrm{ind}(C) = 0 \Rightarrow C$ Jacobi integrable
- Geometric Measure Theory, elliptic regularity $\Rightarrow Y$ CS

Uniqueness

Theorem (Karigiannis–L. 2012)

AC G_2 manifolds $\Lambda^2_+ T^*\mathcal{S}^4$ and $\mathbb{S}(\mathcal{S}^3)$ are locally unique

- Topological data plus $\operatorname{ind}(C)$ measures AC deformations
- L homogeneous $\Rightarrow$ (Moroianu–Semmelmann 2010) representation theory gives $\operatorname{ind}(C) = 0$
- CS G_2 manifolds with $L = \mathbb{CP}^3$ or $L = \mathcal{S}^3 \times \mathcal{S}^3$ have smooth moduli space
- Only topological obstruction to gluing these AC M_- to CS M_+
- $M = M_+ \# t M_- \Rightarrow \dim \mathcal{M}(M_+) = \dim \mathcal{M}(M) - 1$
- $\operatorname{ind}(C) = 8$ for $L = \mathrm{SU}(3)/T^2$