Coassociative conifolds 2

Singularities and stability

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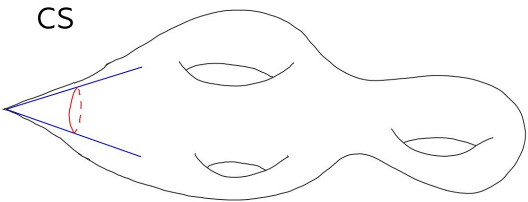
Introduction

Compact coassociative $N^4 \subseteq (M^7, \varphi)$ compact G_2 manifold

- φ nondegenerate harmonic 3-form
- $\rightsquigarrow$ metric g_φ with $\mathrm{Hol}(g_\varphi) \subseteq G_2$
- $\varphi|_N = 0 \Leftrightarrow *_\varphi\varphi|_N = \mathrm{vol}_N$

Conically singular



Aim: understand and use "stability"

Motivation

- SYZ conjecture $\rightsquigarrow$ understand Mirror Symmetry of CY 3-fold using SL fibrations
- "suggests" SL fibrations exist and SLs encodes CY geometry
- "suggests" coassociative fibrations exist and coassociatives encode G_2 geometry
- Coassociative fibrations must have singular fibres
- Constructions of compact G_2 manifolds involve perturbation
- $\rightsquigarrow$ need to understand deformations of singular coassociatives

CS coassociative 4-folds

Definition

N CS if $\exists$ cone $C \cong (0,\infty) \times L$, compact K, $\epsilon > 0$, diffeomorphism $\Phi : (0,\epsilon) \times L \to N \setminus K$ and rate $\lambda \in (1,2)$ such that

$$|\nabla^j(\Phi(r,x) - rx)| = O(r^{\lambda - j}) \quad \text{for all } j \in \mathbb{N} \text{ as } r \to 0$$

- N CS rate $\lambda_0 \rightsquigarrow N$ CS any rate $\lambda \in (1, \lambda_0] \Rightarrow$ choose $\lambda \sim 1$

Question: When is a singular coassociative CS?

Theorem (L. 2012)

N coassociative integral current, tangent cones multiplicity one and Jacobi integrable $\Rightarrow N$ CS

- C Jacobi integrable $\Leftrightarrow$ all infinitesimal deformations of C as coassociative cone are integrable

Deformations

Theorem (L. 2007)

N CS coassociative in $(M, \varphi) \Rightarrow$

- *$\exists$ finite-dimensional $\mathcal{I}$, $\mathcal{O}$*
- *$\exists$ smooth map $\pi : \mathcal{I} \to \mathcal{O}$*

such that moduli space $\mathcal{M}(N) \cong \pi^{-1}(0)$ locally and

$$\dim \mathcal{I} - \dim \mathcal{O} = b^2_+(N) - \dim \operatorname{Im}(H^2(N) \to H^2(L)) - \sum_{\mu \in (-2,1]} m_L(\mu) + \dim \mathcal{C}$$

- $b^2_+(N) = \dim\{\alpha \in L^2(\Lambda^2_+ T^*N) \,:\, \mathrm{d}\alpha = 0\}$
- $m_L(\mu) = \dim\{\gamma \in C^\infty(T^*L) \,:\, *\mathrm{d}\gamma = (\mu+2)\gamma,\ \mathrm{d} * \gamma = 0\}$
- $\mathcal{C}$ is $G_2 \ltimes \mathbb{R}^7$ orbit of C (can generalise to deformation family)

Proof idea

Solve $G(\alpha, \beta) = F(\alpha) + d^*\beta = 0$ for $(\alpha, \beta) \in L^2_{k,\lambda}(\Lambda^2_+ \oplus \Lambda^4)$

- Key difficulty:

$$d(L^2_{k,\lambda}(\Lambda^2_+)) \oplus d^*(L^2_{k,\lambda}(\Lambda^4)) \neq d(L^2_{k,\lambda}(\Lambda^2)) \oplus d^*(L^2_{k,\lambda}(\Lambda^4))$$

- Difference in spaces is $\mathcal{O}$
- $\mathrm{d}G|_0$ not surjective $\Rightarrow$ Implicit Function Theorem does not apply
- Define $H(\alpha, \beta, \gamma) = G(\alpha, \beta) + \gamma$ for $\gamma \in \mathcal{O}$
- $\mathrm{d}H|_0$ surjective $\Rightarrow$ use Implicit Function Theorem
- $H^{-1}(0) = \mathcal{I} \cong \operatorname{Ker} \mathrm{d}G|_0$ smooth
- $G^{-1}(0) = \operatorname{Ker}\{H^{-1}(0) \to \mathcal{O}\}$

Stability index

Proposition (L. 2012)

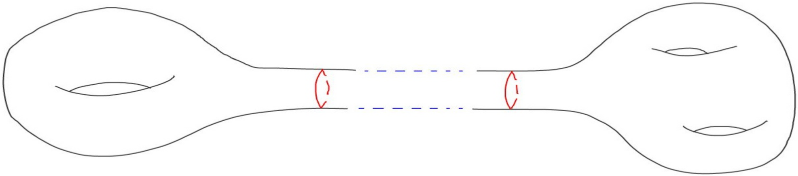
$\dim \mathcal{O} \leq \operatorname{ind}(C) = \sum_{\mu \in (-1,1]} m_L(\mu) - \dim \mathcal{C}$

- $\operatorname{ind}(C)$ stability index of C
- $\operatorname{ind}(C) \geq 0$
- $\operatorname{ind}(C) = 0 \Rightarrow \mathcal{M}(N)$ smooth
- $\operatorname{ind}(C) = 0 \Rightarrow N$ "stable" under deformations of φ, i.e. given φ_s there exists N_s with $\varphi_s|_{N_s} \equiv 0$
- $\operatorname{ind}(C)$ measures obstructions to deforming N

Twisted connected sums

Question: Are there examples of CS coassociative 4-folds?

(Kovalev 2003, CHNP 2012) Take $Z_\pm$ asymptotically cylindrical CY 3-folds



- $Z^6_\pm \sim \mathcal{S}^1 \times \mathbb{R}^+ \times Y^4_\pm$, $Y^4_\pm$ K3 surface
- $\mathcal{S}^1 \times Z^6_\pm \sim \mathcal{S}^1 \times \mathcal{S}^1 \times \mathbb{R}^+ \times Y^4_\pm$
- "Twisted connected sum": swap circles plus hyperkähler rotation $\rightsquigarrow (M^7, \psi)$, $\mathrm{d}\psi = 0$, $\mathrm{d}^*_\psi \psi$ "small"
- (Joyce 1994) Perturb $\psi \rightsquigarrow (M, \varphi)$ holonomy G_2 manifold

Strategy

Fact: $N \subseteq Z_{\pm}$ complex surface $\rightsquigarrow$ N coassociative in (M, ψ)

- N stable $\Rightarrow \exists$ coassociative deformation of N in holonomy G_2 manifold (M, φ)
- Need singular complex surface which is CS and stable
- Need Jacobi integrable tangent cones C with $\mathrm{ind}(C) = 0$

N complex surface with ordinary double point singularities

- $C \cong C_{\mathbf{a}} = \{(0, z_1, z_2, z_3) \in \mathbb{R} \oplus \mathbb{C}^3 \ : \ a_1 z_1^2 + a_2 z_2^2 + a_3 z_3^2 = 0\}$ with $a_1, a_2, a_3 > 0$ and $a_1 + a_2 + a_3 = 3$
- $L_{\mathbf{a}} \cong \mathbb{RP}^3 \subseteq \mathcal{S}^5 \subseteq \mathcal{S}^6$ Hopf lift of degree 2 holomorphic curve $\Sigma \cong \mathbb{CP}^1 \subseteq \mathbb{CP}^2$
- Need $m_{L_{\mathbf{a}}}(\mu)$ for $\mu \in (-1, 1]$

Symmetric case

Want to solve $*\mathrm{d}\gamma = (\mu + 2)\gamma$

Symmetric case: $\mathbf{a} = (1, 1, 1) \Rightarrow L = L_{(1,1,1)} \cong \mathrm{SU}(2)/\mathbb{Z}_2$ with left-invariant metric

- Write $\gamma = f_1\gamma_1 + f_2\gamma_2 + f_3\gamma_3$
- Spectrum of $*\mathrm{d}$ related to spectrum of Δ
- $m_L(\mu) = \begin{cases} 7 & \mu = 0 \\ 16 & \mu = 1 \\ 0 & \mu \neq 0, 1 \end{cases}$
- $\mathcal{C}$ deformation family of $C \Rightarrow$ $\dim \mathcal{C} = \dim \mathbb{R}^7 + \dim \mathrm{G}_2 + 2 = 23$
- $\mathrm{ind}(C) = m_L(0) + m_L(1) - \dim \mathcal{C} = 0$

Hopf lifts

L_a Hopf lift of holomorphic curve in $\mathbb{CP}^2$

Theorem (L. 2012)

L Hopf lift of holomorphic curve Σ degree d_Σ in $\mathbb{CP}^2$ $\Rightarrow$

$$m_L(\mu) = \begin{cases} 0 & \mu \in (-1,0) \\ d_\Sigma^2 + d_\Sigma + 1 & \mu = 0 \\ n_L(\mu(\mu+2)) & \mu \in (0,1) \\ d_\Sigma^2 + 3d_\Sigma + n_L(3) & \mu = 1 \end{cases}$$

where $n_L(\nu) = \dim\{f \in C^\infty(L) \,:\, \Delta f = \nu f\}$

- Fourier expansion of γ such that $*\mathrm{d}\gamma = (\mu+2)\gamma$
- $\gamma \leftrightarrow$ eigenfunctions of Δ plus $H^0(K_\Sigma \otimes H^{\mu+2})$

Existence

Lemma

$C_{\mathbf{a}}$ *Jacobi integrable and* $\mathrm{ind}(C_{\mathbf{a}}) = 0$

- $$m_{L_{\mathbf{a}}}(\mu) = \begin{cases} 0 & \mu \in (-1,0) \\ 7 & \mu = 0 \\ n_{L_{\mathbf{a}}}(\mu(\mu+2)) & \mu \in (0,1) \\ 10 + n_{L_{\mathbf{a}}}(3) & \mu = 1 \end{cases}$$
- $\mathrm{ind}(C_{\mathbf{a}}) = (n_{L_{\mathbf{a}}}(3) - 6) + \sum_{\mu \in (0,1)} n_{L_{\mathbf{a}}}(\mu(\mu+2))$
- $n_{L_{\mathbf{a}}}(3) \geq 6 = \mathfrak{su}(3)^{\perp}$
- $m_{L_{\mathbf{a}}}(1) \leq m_{L_{(1,1,1)}}(1) = 16 \Rightarrow n_{L_{\mathbf{a}}}(3) = 6$

Corollary (L. 2012)

$\exists\ N$ *CS coassociative in twisted connected sum* G_2 *manifold*