Introduction to Lagrangian Mean Curvature Flow

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The following notes are based on a series of talks given by Jason Lotay at the KCL/UCL geometric analysis reading seminar in March 2014. The typesetting was done by Tom Begley (T.Begley@maths.cam.ac.uk) and Kim Moore (K.Moore@maths.cam.ac.uk). Corrections are welcomed by either of us.

1 Introduction

We will aim to address the following questions:

- What is Lagrangian mean curvature flow?
- Why study it?
- What do we know so far?
- What don't we know? What are the problems hindering further progress?
- Where do we go next?

To keep things simple we are going to consider $\mathbb{C}^2 = \mathbb{R}^4$, since all of the problems already arise in this relatively simple setting, so there is no point in considering anything more complicated for the moment. We have the usual complex coordinates

$$z_1 = x_1 + iy_1 \qquad z_2 = x_2 + iy_2.$$

We define the 2-form

$$\omega := \mathrm{d}x_1 \wedge \mathrm{d}y_1 + \mathrm{d}x_2 \wedge \mathrm{d}y_2 = \frac{i}{2}(\mathrm{d}z_1 \wedge \mathrm{d}\overline{z_1} + \mathrm{d}z_2 \wedge \mathrm{d}\overline{z_2}).$$

Since the coefficients are constant, we have that ω is closed, i.e. $\mathrm{d}\omega = 0$, and a simple calculation verifies that

$$\frac{\omega \wedge \omega}{2} = \mathrm{vol}_{\mathbb{C}^2} = \mathrm{vol}_{\mathbb{R}^4}$$

which implies that ω is a symplectic form.
Suppose now that $L \subset \mathbb{C}^2$ is a submanifold and $\dim L = 2$. We say L is Lagrangian if $\omega|_L = 0$.

Example. $\mathbb{R}^2 = \{x_1, x_2\}$ is an example of a Lagrangian submanifold. More generally

$$L = \{(x_1, x_2, f_1(x_1, x_2), f_2(x_1, x_2)) | x_1, x_2 \in \mathbb{R}\}$$

is Lagrangian if and only if there exists some g such that $f_1 = \partial_1 g$ and $f_2 = \partial_2 g$, since the condition $\omega|_L = 0$ is equivalent to $\partial_1 f_2 = \partial_2 f_1$.

On $\mathbb{C}^2$ we have a complex structure J defined by the relations

$$J\frac{\partial}{\partial x_j} = \frac{\partial}{\partial y_j} \qquad J\frac{\partial}{\partial y_j} = -\frac{\partial}{\partial x_j}$$

A simple calculation shows that this complex structure relates ω to the standard inner product on $\mathbb{C}^4$ via

$$\omega(X, Y) = g(JX, Y)$$

Indeed if

$$X = X^1\frac{\partial}{\partial x_1} + X^2\frac{\partial}{\partial x_2} + X^3\frac{\partial}{\partial y_1} + X^4\frac{\partial}{\partial y_2}$$
$$Y = Y^1\frac{\partial}{\partial x_1} + Y^2\frac{\partial}{\partial x_2} + Y^3\frac{\partial}{\partial y_1} + Y^4\frac{\partial}{\partial y_2}$$

then

$$JX = X^1\frac{\partial}{\partial y_1} + X^2\frac{\partial}{\partial y_2} - X^3\frac{\partial}{\partial x_1} - X^4\frac{\partial}{\partial x_2}$$

so

$$g(JX, Y) = X^1Y^3 + X^2Y^4 - X^3Y^1 - X^4Y^2.$$

On the other hand

$$\omega(X, Y) = \det\begin{pmatrix} X^1 & Y^1 \\ X^3 & Y^3 \end{pmatrix} + \det\begin{pmatrix} X^2 & Y^2 \\ X^4 & Y^4 \end{pmatrix} = X^1Y^3 - X^3Y^1 + X^2Y^4 - X^4Y^2,$$

so equality is established.

2 Minimal Surfaces

We know from minimal surfaces that the condition $H = 0$ (where H is mean curvature) is equivalent to being stationary for the area functional

$$\mathcal{A}(L) = \int_L \mathrm{vol}_L. \tag{2.1}$$

Minimal surfaces need not actually be area minimising, as examples like the catenoid show. However, minimal Lagrangians turn out to be area minimising. The proof is by a calibration argument similar to that used to show that minimal graphs are in fact area minimising. The key point is that $\Omega := \mathrm{d}z_1 \wedge \mathrm{d}z_2$ is preserved by $\mathrm{SU}(2) \subset \mathrm{SO}(4)$

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \middle| a, b \in \mathbb{C} \quad |a|^2 + |b|^2 = 1 \right\}.$$

Identification with a subset of SO(4) comes from identifying complex numbers with 2×2 matrices

$$c + id = \begin{pmatrix} c & -d \\ d & c \end{pmatrix}.$$

Suppose then that $z' = Az$ with $A \in \mathrm{SU}(2)$, then

$$\begin{aligned} \mathrm{d}z'_1 \wedge \mathrm{d}z'_2 &= a_{1j}\mathrm{d}z_j \wedge a_{2k}\mathrm{d}z_k \\ &= (a_{11}a_{22} - a_{12}a_{21})\mathrm{d}z_1 \wedge \mathrm{d}z_2 \\ &= (a\overline{a} - b(-\overline{b}))\mathrm{d}z_1 \wedge \mathrm{d}z_2 \\ &= \mathrm{d}z_1 \wedge \mathrm{d}z_2. \end{aligned}$$

Moreover, J is preserved by U(2)

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} \middle| a, b \in \mathbb{C} \right\}.$$

so it follows that Lagrangian submanifolds are preserved by U(2). Hence any Lagrangian plane can be seen as $\mathbb{R}^2$ via U(2) transformations.
Suppose that L is a Lagrangian plane with orthonormal basis $\{\underline{e}_1, \underline{e}_2\}$. Use SU(2) to rotate L so that $\underline{e}_1 = \partial/\partial x_1$. Then as $\underline{e}_1$ and $\underline{e}_2$ are orthogonal we have that

$$\underline{e}_2 = a_1 \frac{\partial}{\partial y_1} + a_2 \frac{\partial}{\partial x_2} + a_3 \frac{\partial}{\partial y_2}.$$

However since $0 = \omega(\underline{e}_1, \underline{e}_2) = a_1$ we get

$$\underline{e}_2 = \cos\theta \frac{\partial}{\partial x_2} + \sin\theta \frac{\partial}{\partial x_2},$$

since $\underline{e}_2$ is unit length. Since

$$\Omega = (\mathrm{d}x_1 + i\mathrm{d}y_1) \wedge (\mathrm{d}x_2 + i\mathrm{d}y_2) = (\mathrm{d}x_1 \wedge \mathrm{d}x_2 - \mathrm{d}y_1 \wedge \mathrm{d}y_2) + i(\mathrm{d}x_1 \wedge \mathrm{d}y_2 + \mathrm{d}x_2 \wedge \mathrm{d}y_1),$$

we calculate

$$\Omega(\underline{e}_1, \underline{e}_2) = \cos\theta + \sin\theta = e^{i\theta}.$$

Hence

$$\Omega_L = e^{i\theta}\mathrm{vol}_L = e^{i\theta}\underline{e}_1^* \wedge \underline{e}_2^*$$

for any Lagrangian plane, where $\underline{e}_i^*$ denotes the dual basis. This is also true for any Lagrangian L, but with θ varying on L. We call θ the Lagrangian angle. Notice that θ is only determined up to multiples of 2π, however we can use it to define the so called phase function $e^{i\theta} : L \to S^1$. There is an important link between the Lagrangian angle and the mean curvature which we will use in our study of Lagrangian mean curvature flow.

Lemma 2.1. *For a Lagrangian submanifold, $\vec{H} = J\nabla\theta$.*

Remark. Notice that multiples of 2π in θ are killed by ∇ so there is no ill-definition here.

Proof. Let $p \in L$ and let $\{\underline{e}_1, \underline{e}_2\}$ be an orthonormal basis for T_pL. Then it follows since L is Lagrangian that $\{\underline{e}_1, \underline{e}_2, J\underline{e}_1, J\underline{e}_2\}$ is an orthonormal basis for $\mathbb{C}^2$. Indeed since $g(J\underline{e}_i, \underline{e}_j) = \omega(\underline{e}_i, \underline{e}_j) = 0$ and using the fact that $\underline{e}_1$ and $\underline{e}_2$ are orthogonal it's easy to check everything works out. Hence we can write

$$\Omega = e^{i\theta}(\underline{e}_1^* + iJ\underline{e}_1^*) \wedge (\underline{e}_2^* + iJ\underline{e}_2^*)$$

(where by a slight abuse of notation we write $J\underline{e}_i^* = (J\underline{e}_i)^*$). Now, since Ω has constant coefficients, we know that for any X we have

$$\overline{\nabla}_X \Omega = 0,$$

hence

$$0 = i\mathrm{d}\theta(X)\Omega + e^{i\theta}(\overline{\nabla}_X \underline{e}_1^* + i\overline{\nabla}_X J\underline{e}_1^*) \wedge (\underline{e}_2^* + iJ\underline{e}_2^*) + e^{i\theta}(\underline{e}_1^* + iJ\underline{e}_1^*) \wedge (\overline{\nabla}_X \underline{e}_2^* + i\overline{\nabla}_X J\underline{e}_2^*).$$

Since $\overline{\nabla}_X \underline{e}_j^* + i\overline{\nabla}_X J\underline{e}_j^*$ must be a scalar multiple of $\underline{e}_j^* + iJ\underline{e}_j^*$, we can rewrite the above as

$$0 = (i\mathrm{d}\theta(X) + e^{i\theta}(\overline{\nabla}_X \underline{e}_1^* + i\overline{\nabla}_X J\underline{e}_1^*)((\underline{e}_1 - iJ\underline{e}_1)/2) + e^{i\theta}(\overline{\nabla}_X \underline{e}_2^* + i\overline{\nabla}_X J\underline{e}_2^*)((\underline{e}_2 - iJ\underline{e}_2)/2))\Omega$$

since $(\underline{e}_1^* + iJ\underline{e}_1^*)((\underline{e}_1 - iJ\underline{e}_1)/2) = 1$ we have

$$(\overline{\nabla}_X \underline{e}_1^* + i\overline{\nabla}_X J\underline{e}_1^*)((\underline{e}_1 - iJ\underline{e}_1)/2) = -(\underline{e}_1^* + iJ\underline{e}_1^*)((\overline{\nabla}_X \underline{e}_1 - i\overline{\nabla}_X J\underline{e}_1)/2).$$

Now, $\overline{\nabla}_X(J\underline{e}_1) = J\overline{\nabla}_X \underline{e}_1$, thus at p we have

$$\overline{\nabla}_X \underline{e}_1 = g(\overline{\nabla}_X \underline{e}_1, J\underline{e}_1)Je_1 + g(\overline{\nabla}_X \underline{e}_1, J\underline{e}_2)J\underline{e}_2,$$

since $\nabla_X \underline{e}_1 = 0$ as $\nabla_{\underline{e}_j}\underline{e}_k = 0$ at p. Thus $\underline{e}_1^*(\overline{\nabla}_X \underline{e}_1) = 0$, since

$$\underline{e}_1^*(\overline{\nabla}_X J\underline{e}_1) = \underline{e}_1^*(J\overline{\nabla}_X \underline{e}_1) = -g(\overline{\nabla}_X \underline{e}_1, J\underline{e}_1).$$

Moreover $J\underline{e}_1^*(\overline{\nabla}_X \underline{e}_1) = g(\overline{\nabla}_X \underline{e}_1, J\underline{e}_1)$. This all together implies

$$(\overline{\nabla}_X \underline{e}_1^* + i\overline{\nabla}_X J\underline{e}_1^*)((\underline{e}_1 - iJ\underline{e}_1)/2) = -ig(\overline{\nabla}_X \underline{e}_1, J\underline{e}_1)$$

thus

$$\begin{aligned}
\mathrm{d}\theta(X) &= g(\overline{\nabla}_X \underline{e}_1, J\underline{e}_1) + g(\overline{\nabla}_x \underline{e}_2, J\underline{e}_2) \\
&= g(\overline{\nabla}_{\underline{e}_1} X, J\underline{e}_2) + g(\overline{\nabla}_{\underline{e}_2} X, J\underline{e}_2) \\
&= -g(X, \overline{\nabla}_{\underline{e}_1}(J\underline{e}_1) + \overline{\nabla}_{\underline{e}_2}(J\underline{e}_2)) = -g(X, J\vec{H})
\end{aligned}$$

□

From the above lemma we see that a Lagrangian L is minimal if and only if θ is constant. Suppose now that $\theta = 0$, then $\Omega_L = \mathrm{vol}_L$ which implies $\mathrm{Re}\Omega_L = \mathrm{vol}_L$ and $\mathrm{Im}\Omega_L = 0$. Now more generally

$$\mathrm{Re}\Omega = \mathrm{d}x_1 \wedge \mathrm{d}x_2 - \mathrm{d}y_1 \wedge \mathrm{d}y_2.$$

This has constant coefficients and so is closed, and $\mathrm{Re}\Omega(\underline{e}_1, \underline{e}_2) \leq 1$ with equality if and only if $\mathrm{Re}\Omega = \mathrm{vol}_L$. Suppose that $L' \in [L]$ is homologous to L, then by Stokes' theorem

$$\mathcal{A}(L') = \int_{L'} \mathrm{vol}_{L'} \geq \int_{L'} \mathrm{Re}\Omega = \int_L \mathrm{Re}\Omega = \int_L \mathrm{vol}_L = \mathcal{A}(L).$$

Question: Given a homology class $[L]$, does there exists a special (that is, minimal) Lagrangian representative?
Answer: We have no idea, but the hope is that Lagrangian mean curvature flow should help us answer this.

3 Minimization Problem

Schoen-Wolfson found $L^2 \subset M^4$ a Calabi-Yau manifold such that a minimiser for area among Lagrangians in $[L]$ exists, is smooth except for finitely many points, but it need not be minimal. Wolfson found $L = S^2 \subset M^4 = K3$ with $[L] \neq 0$ such that the Lagrangian minimising area amongst Lagrangians in $[L]$ exists and is not minimal, and the surface minimising area in $[L]$ exists and is not Lagrangian.
Neves showed that given L^2 embedded Lagrangian in M^4 Calabi-Yau, then there exists a Lagrangian L' Hamiltonian isotopic to L such that hte Lagrangian mean curvature flow starting at L' has a finite time singularity.

4 Examples

1. As we saw before the submanifold defined by

$$L = \{(x_1, f_1(x_1, x_2), x_2, f_2(x_1, x_2)) | (x_1, x_2) \in \mathbb{R}\}$$

is Lagrangian precisely when $\partial_1 f_2 = \partial_2 f_1$. In particular if there is some function g such that $f_1 = \partial_1 g$ and $f_2 = \partial_2 g$ then L is Lagrangian. If we take $g(x_1, x_2) = \log(x_1^2 + x_2^2)/2$ on $\mathbb{R}^2 \setminus \{0\}$. Then L is given by

$$\left\{ \left(x_1, \frac{x_1}{x_1^2 + x_2^2}, x_2, \frac{x_2}{x_1^2 + x_2^2} \right) \middle| \, x_1, x_2 \neq 0 \right\}.$$

This is known as the Lagrangian catenoid. We know that mean curvature is given by $\vec{H} = J\nabla\theta$, how do we calculate this? A basis for the tangent space of L is given by

$$\begin{aligned} \underline{e}_1 &= \left(1, \frac{x_2^2 - x_1^2}{(x_1^2 + x_2^2)^2}, 0, \frac{-2x_1 x_2}{(x_1^2 + x_2^2)^2} \right) \\ \underline{e}_2 &= \left(0, \frac{-2x_1 x_2}{(x_1^2 + x_2^2)^2}, 1, \frac{x_1^2 - x_2^2}{(x_1^2 + x_2^2)^2} \right). \end{aligned}$$

Recalling that the holomorphic volume form is given by

$$\Omega = \mathrm{d}z_1 \wedge \mathrm{d}z_2 = \mathrm{d}x_1 \wedge \mathrm{d}x_2 - \mathrm{d}y_1 \wedge \mathrm{d}y_2 + i(\mathrm{d}x_1 \wedge \mathrm{d}y_2 + \mathrm{d}y_1 \wedge \mathrm{d}x_2)$$

we can see that

$$\mathrm{Im}\Omega(\underline{e}_1, \underline{e}_2) = \frac{x_1^2 - x_2^2}{(x_1^2 + x_2^2)^2} + \frac{x_2^2 - x_1^2}{(x_1^2 + x_2^2)^2} = 0, \tag{4.1}$$

hence, since we already know $\Omega|_L = e^{i\theta}\mathrm{vol}|_L$, it follows that $\Omega|_L = \pm\mathrm{vol}|_L$, choosing the orientation appropriately we can ensure $\Omega|_L = \mathrm{vol}|_L$. This means $\theta \equiv 0$ (up to multiples of 2π and so $\nabla\theta = 0$

and so $\vec{H} = 0$.
2. Consider the submanifold of $\mathbb{C}^2$ given by

$$L = \{(e^{i\theta_1}, e^{i\theta_2}) | \theta_1, \theta_2 \in \mathbb{R}\}$$

This is Lagrangian. Considering the tangent vectors $\underset{\sim}{e}_1 = (ie^{i\theta_1}, 0)$ and $\underset{\sim}{e}_2 = (0, ie^{i\theta_2})$ (which form a basis for the tangent space of L we get

$$\Omega(\underset{\sim}{e}_1, \underset{\sim}{e}_2) = -e^{i(\theta_1+\theta_2)}$$

which implies that the Lagrangian angle is given by $\theta = \theta_1 + \theta_2 + \pi$. This in particular shows that the Maslov class of $\mathrm{d}\theta$ is non-zero. Since minimal Lagrangians have zero Maslov class, and the Maslov class doesn't change under Lagrangian mean curvature flow, if we hope to get convergence to a special Lagrangian by flowing a Lagrangian manifold, it stands to reason that we should restrict our attention to those with zero Maslov class. In the above example one can compute that $\vec{H} = -\vec{F}^{\perp}$, so this is a self-shrinking solution.
3. Define

$$L := \left\{ \frac{(u_1, u_1u_3, u_2, u_2u_3)}{1+u_3^2} \middle| u_1^2 + u_2^2 + u_3^2 = 1 \right\}.$$

This is known as the Whitney sphere and is an immersed S^2. It can be seen as a compactification of the Lagrangian catenoid. It is a Willmore surface and minimises the Willmore functional in its homotopy class. It is the only genus zero Lagrangian Willmore surface. There are no embedded Lagrangian S^2 in $\mathbb{C}^2$.
Let $\lambda = x_1\mathrm{d}y_1 + x_2\mathrm{d}y_2 - y_1\mathrm{d}x_1 - y_2\mathrm{d}x_2$, then one can check that $\mathrm{d}\lambda = 2\omega$. We say a Lagrangian L is exact if λ is exact, i.e. if $\lambda|_L = \mathrm{d}\beta$.

Claim: There are no non-trivial, smooth, zero Maslov class self-shrinkers.

Lemma 4.1. *A Lagrangian self-shrinker/expander L is zero Maslov class if and only if it is exact.*

Proof. Suppose that $\vec{H} = \kappa\vec{F}^{\perp}$ where $\kappa \neq 0$ and suppose that L has zero Maslov class. Then

$$J\nabla\theta = \kappa\vec{F}^{\perp} \iff \nabla\theta = -\kappa J\vec{F}^{\perp} = -\kappa(J\vec{F})^{\perp}$$

since for a Lagrangian $\omega|_L = 0$, so $\omega(X,Y) = g(JX,Y)$. So $J(T_pL) = (T_pL)^{\perp}$. Hence

$$-\kappa(J\vec{F})^T = -\kappa(-y_1, x_1, -y_2, x_2)^T$$

so

$$\mathrm{d}\theta = -\kappa\lambda|_L$$

which implies L is exact if it has zero Maslov class. $\square$

Theorem 4.2 (Gromov)**.** *There are no compact embedded exact Lagrangians in $\mathbb{C}^2$.*

Corollary 4.3. *There are no compact embedded zero Maslov class self-shrinkers.*

Theorem 4.4 (Smoczyk)**.** *For compact L, Lagrangian mean curvature flow exists in Calabi-Yau manifolds (Ricci-flat and Kähler). Can even relax this to Kähler-Einstein.*

The following is *not* a proof. Using Cartan's formula we find

$$\frac{\partial \omega}{\partial t} = \mathcal{L}_{\vec{H}}\omega = \mathrm{d}(\vec{H} \lrcorner \omega) = \mathrm{d}(\mathrm{d}\theta) = 0$$

since if $\vec{H} = J\nabla\theta$ then $\omega(\vec{H}, \cdot) = g(J\vec{H}, \cdot) = -\mathrm{d}\theta(\cdot)$. However for this we are using the fact that L stays Lagrangian under the flow, which is precisely what we are trying to establish! We need to find a different approach.

Proof. For a compact submanifold one can run a fixed point argument to get short time existence of flow by mean curvature by writing everything as a normal graph over the initial condition (very roughly speaking anyway). What we are really interested in showing is that the submanifolds stay Lagrangian under the flow. This is done using the maximum principle. Define the function $f := |\omega_t|^2$ (to keep notation simple we adopt the convention that whenever a subscript t appears we are considering the restriction to L_t). We want to show that f satisfies a differential inequality of the form

$$\frac{\partial f}{\partial t} \leq -\nabla^*\nabla f + Bf$$

so that we may apply the maximum principle. The trick is to first show that a small pertubation of a Lagrangian is 'totally real', that is to say $J(T_pL) \cap T_pL = \{0\}$. Hence flowing for a short time the submanifolds may not remain Lagrangian, but we can assume this property. We now compute, using normal coordinates at a point and considering a tangential vector field X

$$\begin{aligned}
\omega(\vec{H}, X) &= \omega(\overline{\nabla}_{\underline{e}_i}\underline{e}_i, X) \\
&= \overline{\nabla}_{\underline{e}_i}(\omega(\underline{e}_i, X)) - \omega(\underline{e}_i, \overline{\nabla}_{\underline{e}_i}X) \\
&= \nabla_{\underline{e}_i}(\omega(\underline{e}_i, X)) - \omega(\underline{e}_i, A(e_i, X)) \\
&= -\mathrm{d}^*\omega(X) \underbrace{-\omega(\underline{e}_i, A(\underline{e}_i, X))}_{=:\xi}.
\end{aligned}$$

We notice that the right hand side is tensorial, so our choice of coordinates was no restriction. We want to compute $\mathrm{d}(\omega(\vec{H}, \cdot))(X, Y)$. Applying d to each of the terms in the right hand side above in turn we find first that

$$\mathrm{d}\mathrm{d}^*\omega|_L = \Delta_d\omega|_L = \nabla^*\nabla\omega|_L + C \lrcorner \omega$$

where C depends on the Riemann curvature of L and other intrinsic quantities only. For the second term suppose that $[X, Y] = 0$ at p, then

$$\begin{aligned}
\mathrm{d}\xi(X, Y) &= X(\xi(Y)) - Y(\xi(X)) \\
&= -\overline{\nabla}_X(\omega(\underline{e}_i, A(\underline{e}_i, Y))) + \overline{\nabla}_Y(\omega(\underline{e}_i, A(\underline{e}_i, X))) \\
&= -2\omega(A(X, \underline{e}_i), A(Y, \underline{e}_i)) - \omega(\underline{e}_i, \overline{\nabla}_X(\overline{\nabla}_Y\underline{e}_i - \nabla_Y\underline{e}_i)) + \omega(\underline{e}_i, \overline{\nabla}_Y(\overline{\nabla}_X\underline{e}_i - \nabla_X\underline{e}_i)) \\
&= -2\omega(A(X, \underline{e}_i), A(Y, \underline{e}_i)) + \omega(\underline{e}_i, R(X, Y)\underline{e}_i) + \omega(\underline{e}_i, A(X, \nabla_Y\underline{e}_i)) - \omega(\underline{e}_i, A(Y, \nabla_X\underline{e}_i)) \\
&= \omega(\underline{e}_i, R(X, Y)\underline{e}_i) - 2\omega(A(X, \underline{e}_i), A(Y, \underline{e}_i)).
\end{aligned}$$

Now, on a Lagrangian submanifold we had that if Z, W are normal then

$$\omega(Z, W) = \omega(JX, JY) = \omega(X, Y).$$

On a totally real submanifold the same need not be true, however we do at least have that if $Z = (JX)^{\perp}$ and $W = (JY)^{\perp}$ then

$$\begin{aligned}
\omega(Z, W) &= \omega(JX - (JX)^T, JY - (JY)^T) \\
&= \omega(JX, JY) - \omega((JX)^T, JY) - \omega(JX, (JY)^T) + \omega((JX)^T, (JY)^T) \\
&= \omega(X, Y) - g(J(JX)^T, JY) + g(X, (JY)^T) + \omega((JX)^{\perp}, (JY)^{\perp}) \\
&= \omega(X, Y) - g(JX, Y) + g(X, JY) + \omega((JX)^{\perp}, (JY)^{\perp}) \\
&= \omega(JX)^{\perp}, (JY)^{\perp})) - \omega(X, Y).
\end{aligned}$$

hence we can conclude that $\mathrm{d}\xi = C \lrcorner \omega|_L$, where C again depends only on intrinsic quantities like the Riemann curvature tensor. Assembling the above we have

$$\mathrm{d}(\vec{H} \lrcorner \omega) = -\nabla^* \nabla \omega|_L + C \lrcorner \omega|_L$$

Now we return to the function $f = |\omega_t|^2 = g_t(\omega_t, \omega_t)$ for $t \in [0, T]$. Then

$$\begin{aligned}
\frac{\partial f}{\partial t} &= \frac{\partial g_t}{\partial t}(\omega_t, \omega_t) + 2g_t\left(\underbrace{\frac{\partial \omega_t}{\partial t}}_{=\mathrm{d}(\vec{H} \lrcorner \omega_t)}, \omega_t\right) \\
&\leq \frac{\partial g_t}{\partial t}(\omega_t, \omega_t) + 2g_t(-\nabla_t^* \nabla_t \omega_t, \omega_t) + C g_t(\omega_t, \omega_t) \\
&\leq -\nabla_t^* \nabla_t \underbrace{g_t(\omega_t, \omega_t)}_{=f} - 2g_t(\nabla_t \omega_t, \nabla_t \omega_t) + Cf \\
&\leq -\nabla_t^* \nabla_t f + Cf.
\end{aligned}$$

This means we can now apply the maximum principle to show that f must remain 0 for all time. To do this we fix $\varepsilon > 0$ and define the new function

$$f_\varepsilon := f - \varepsilon e^{2Ct}.$$

Then since $f(0) = 0$ we have $f_\varepsilon(0) = -\varepsilon < 0$. Moreover, if we differentiate f_ε in t we get

$$\begin{aligned}
\frac{\partial f_\varepsilon}{\partial t} &\leq -\nabla_t^* \nabla_t f + Cf - 2C\varepsilon e^{2Ct} \\
&= -\nabla_t^* \nabla_t f_\varepsilon + Cf_\varepsilon - C\varepsilon e^{2Ct} \\
&< -\nabla_t^* \nabla_t f_\varepsilon + Cf_\varepsilon
\end{aligned}$$

for all $t \geq 0$. Suppose that there exists $y \in L$ and $T_0 \in (0, T]$ such that $f_\varepsilon(y, T_0) = -\varepsilon/2$, where T_0 is the first time f_ε attains this value. Then

$$\frac{\partial f_\varepsilon}{\partial t} \geq 0 \qquad -\nabla_t^* \nabla_t f_\varepsilon(y, T_0) \leq 0$$

but

$$0 \leq \frac{\partial f_\varepsilon}{\partial t} < -\nabla_t^* \nabla_t f_\varepsilon + Cf_\varepsilon \leq 0,$$

a contradiction. Hence letting $\varepsilon \to 0$ we see that $f \equiv 0$ for all time. $\square$

Lemma 4.5. *The evolution of θ (for zero Maslov class) is given by the equation*

$$\frac{\partial\theta}{\partial t} = -\mathrm{d}^*\mathrm{d}\theta = -\nabla^*\nabla\theta = \Delta\theta.$$

Proof. We have

$$\frac{\partial\Omega}{\partial t} = \mathcal{L}_{\vec{H}}\Omega = \mathrm{d}(\vec{H}\lrcorner\Omega) = \mathrm{d}(e^{i\theta}i\nabla\theta\lrcorner\mathrm{vol}|_L) = -e^{i\theta}\mathrm{d}\theta\wedge *\mathrm{d}\theta + ie^{i\theta}\mathrm{d}(*\mathrm{d}\theta).$$

On the other hand

$$\frac{\partial\Omega}{\partial t} = \frac{\partial}{\partial t}(e^{i\theta}\mathrm{vol}|_L) = ie^{i\theta}\frac{\partial\theta}{\partial t}\mathrm{vol}|_L + e^{i\theta}\frac{\partial}{\partial t}(\mathrm{vol}|_L).$$

Comparing real and imaginary parts we get

$$\frac{\partial\theta}{\partial t} = -\mathrm{d}^*\mathrm{d}\theta.$$

□

Suppose now that L is a smooth zero Maslov class self-shrinker, i.e. that $L_t = \sqrt{-t}L$ for all $t < 0$. Let

$$\rho(t) := \int_{L_t}\theta_t^2\Phi(0,0)\mathrm{d}\mathcal{H}^2,$$

where

$$\Phi(x_0, T) = \frac{e^{-|x-x_0|^2/4(T-t_0)}}{4\pi(T-t)}.$$

Then Huisken's monotonicity formula says that

$$0 = \frac{\partial\rho}{\partial t} = \int_{L_t}\left(\frac{\partial\theta_t^2}{\partial t} - \Delta\theta_t^2\right)\Phi(0,0)\mathrm{d}\mathcal{H}^2 - \int_{L_t}\left|\vec{H} - \frac{\vec{F}}{2t}\right|^2\Phi(0,0)\mathrm{d}\mathcal{H}^2.$$

Now the evolution equation for θ says that

$$\frac{\partial\theta^2}{\partial t} = \Delta\theta^2 - 2|\mathrm{d}\theta|^2$$

so

$$\frac{\partial\theta^2}{\partial t} - \Delta\theta^2 = -2|\mathrm{d}\theta|^2 \leq 0$$

Hence $\partial\rho/\partial t = 0$ implies $\mathrm{d}\theta = 0$ which implies L is minimal. So $0 = \vec{H} = \vec{F}^\perp$ so L is a cone, and so since it is smooth, L is in fact a plane.