Uniqueness of Lagrangian self-expanders

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Outline

- **Lagrangian Mean Curvature Flow**
- **Lagrangian self-expanders**
- **Uniqueness result**
- **Sketch proof**

Lagrangian Mean Curvature Flow

Definition

$\mathbf{x}_t : L \hookrightarrow M \rightsquigarrow$ *Mean Curvature Flow* $\dot{\mathbf{x}}_t = H(\mathbf{x}_t)$.

- MCF is the gradient flow for the area functional.
- Stationary points are minimal submanifolds.

Theorem (Smoczyk 1996)

M^{2n} *Calabi–Yau* , $\mathbf{x}_t : L^n \hookrightarrow M^{2n}$ *satisfies MCF,*
$\mathbf{x}_0^*(\omega) = 0 \Rightarrow \mathbf{x}_t^*(\omega) = 0 \ \forall t \rightsquigarrow$ *Lagrangian Mean Curvature Flow.*

- Stationary points of LMCF are special Lagrangian.
- Compact SL are area-minimizing in their homology class.

Finding special Lagrangians

Question

Given Lagrangian L , is there SL $L' \in [L]$? Is L' unique?

- (Schoen–Wolfson 2001) Minimizing compact Lagrangian L^2 in class exist but may not be SL.
- (Wolfson 2005) $\exists\ L \cong \mathcal{S}^2$ such that minimizing Lagrangian in $[L]$ not SL and minimizer in $[L]$ exists but not Lagrangian.

Conjecture (Thomas–Yau 2002)

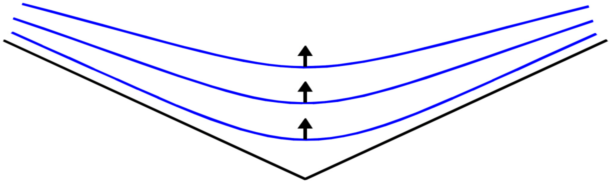
L compact is "stable" $\Rightarrow$ LMCF converges to unique SL in $[L]$.

- (Neves 2011) L^2 compact $\Rightarrow \exists\ L' \sim L$ such that LMCF develops finite-time singularity.

Singularities and self-expanders

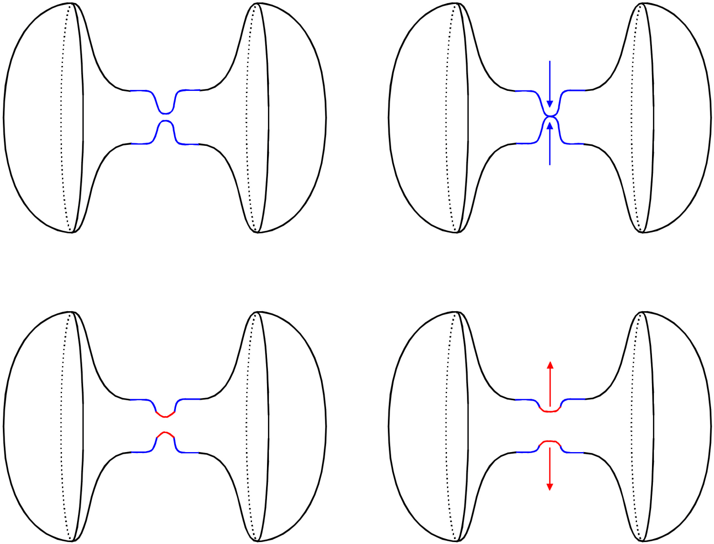
Definition

$\mathbf{x} : L^n \to \mathbb{C}^n$ *self-expander* if $H = \mathbf{x}^\perp \rightsquigarrow \mathbf{x}_t = \sqrt{2t}\mathbf{x}$ solves LMCF.



- Self-expanders are stationary for weighted area functional.
- (Neves–Tian 2007) Blow-downs of eternal solutions to LMCF are self-expanders for positive time.
- Self-expanders solve LMCF with singular initial condition $\rightsquigarrow$ potential surgery for LMCF.

Surgery



Examples

- (Anciaux 2006) SO(n)-invariant examples $\rightsquigarrow L \cong \mathcal{S}^{n-1} \times \mathbb{R}$ asymptotic to transverse SO(n)-invariant pair of planes.
- (Lee & Wang 2007) Hamiltonian stationary examples in $\mathbb{C}^n$.
- (Joyce–Lee–Tsui 2008) Generalised all known examples , including L asymptotic to any transverse pair of planes.
- (Castro–Lerma 2009) Classification of Hamiltonian stationary self-expanders in $\mathbb{C}^2$.
- (Chau–Chen–He 2009) $\exists$ 1-1 correspondence between cones $L_0 = \{x + J\nabla\psi_0(x)\}$ and self-expanders $L = \{x + J\nabla\psi(x)\}$ asymptotic to L_0 with ψ_0, ψ satisfying a Hessian bound.

Planar ends

Question

Can we classify Lagrangian self-expanders with two planar ends?

- (Schoen 1983) Catenoid is unique minimal hypersurface with two planar ends.
- In $\mathbb{C}^2$ SL surfaces are holomorphic curves after hyperkähler rotation $\rightsquigarrow$ classification.
- Classification of SL in $\mathbb{C}^n$ with two planar ends is not known for $n > 2$.
- (Ilmanen–White) Self-expanders in $\mathbb{C}$ are geodesics for metric with non-positive curvature $\rightsquigarrow$ uniqueness.
- (Nakahara 2011) Families of singular Lagrangian self-expanders with the same two planar ends in $\mathbb{C}^n$.

Uniqueness result

Theorem (L–Neves)

Zero Maslov class Lagrangian self-expanders asymptotic to transverse pairs of planes in $\mathbb{C}^n$ are

- *locally unique if $n > 2$;*
- *unique if $n = 2$.*

Definition

$\Omega = \mathrm{d}z_1 \wedge \ldots \mathrm{d}z_n$, *L Lagrangian* $\rightsquigarrow \Omega|_L = e^{i\theta}\,\mathrm{vol}_L$
$\rightsquigarrow$ *Lagrangian angle* θ.

- L zero Maslov class if θ single-valued function.
- L Hamiltonian stationary if $\Delta\theta = 0$.
- Key fact: $H = J\nabla\theta$.

Strategy

$L \subseteq \mathbb{C}^2$ zero Maslov class self-expander asymptotic to P.

Lemma

P SO(2)-invariant $\Rightarrow$ L SO(2)-invariant $\Rightarrow$ L unique.

Proof: L SO(2)-invariant $\Leftrightarrow \mu = x_1 y_2 - y_1 x_2 = 0$ on L.

Monotonicity Formula: $\frac{\mathrm{d}}{\mathrm{d}t}\mu^2 = \Delta\mu^2 - 2|\nabla\mu|^2$
$\mu = 0$ at $t = 0 \Rightarrow \mu = 0$ for $t > 0$.

- Choose $P(s)$ such that $P(0) = P$, $P(1)$ is SO(2)-invariant.
- $\mathcal{S} = \{$zero Maslov class self-expanders asymptotic to $P(s)\}$.
- Deformation theory $\Rightarrow \pi : \mathcal{S} \to [0, 1]$ local diffeomorphism.
- Show $\mathcal{S}$ is compact $\Rightarrow \pi$ covering map.
- $\pi^{-1}(1)$ one element $\Rightarrow \pi$ diffeomorphism $\Rightarrow$ uniqueness.

Local uniqueness

Liouville form $\lambda = \sum_{j=1}^{n} x_j \mathrm{d}y_j - y_j \mathrm{d}x_j \Rightarrow \mathrm{d}\lambda = 2\omega$.

Definition

L is exact if $\lambda|_L = \mathrm{d}\beta$.

Lemma

Zero Maslov class Lagrangian self-expanders are exact.

Proof: $H = J\nabla\theta = \mathbf{x}^{\perp} \Leftrightarrow \nabla\theta = -J\mathbf{x}^{\perp} = -(J\mathbf{x})^{\top} \Leftrightarrow$
$\lambda|_L = -\mathrm{d}\theta = \mathrm{d}\beta$.

- Exact zero Maslov class L' near $L \leftrightarrow$ graphs L_u of $J\nabla u$.
- L_u self-expander $\Leftrightarrow F(u) = \beta_u + \theta_u$ constant.
- $\mathrm{d}F|_0(u) = \Delta u + \langle \mathbf{x}, \nabla u\rangle - 2u \Rightarrow \mathrm{d}F|_0$ isomorphism.
- Inverse Function Theorem $\Rightarrow$ local uniqueness.

Compactness

Let L^j be zero Maslov class self-expanders asymptotic to P^j.

- (Ilmanen) L^j has subsequence converging to (possibly singular) self-expander L.
- L asymptotic to transverse pair of planes P.
- Suffices to show L is smooth self-expander.

Lemma

L is not minimal.

Proof: Suppose $H = 0 = \mathbf{x}^\perp \Rightarrow L$ is a cone $\Rightarrow L = P$.

- (Neves) $\beta^j \to \beta$ constant $\Rightarrow \theta^j = -\beta^j \to -\beta$ constant.
- $\rightsquigarrow P$ is SL $\rightsquigarrow$ contradicts P^j not SL.

Gaussian density

Define Gaussian density

$$\Theta(y,l) = \int_L (4\pi l)^{-\frac{n}{2}} \exp\left(-\frac{|x-y|^2}{4l}\right) \mathrm{d}\mathcal{H}^n$$

Lemma

Given $\epsilon > 0$, $\exists \delta > 0$ such that $\Theta(y,l) < 1+\epsilon$ for all $l \leq \delta$.

Proof: (Huisken) Monotonicity Formula $\Rightarrow \Theta(y,l) \leq 2$.

- $\Theta(y,l) = 2 \Rightarrow L$ self-shrinker $\Rightarrow L$ minimal $\Rightarrow$ contradiction.

Suppose that $\Theta(y_k,\delta_k) \geq 1+\epsilon$ for $\delta_k \to 0$.

- $\rightsquigarrow$ blow-up L' of L which is union of SLs and not a plane.
- Blow-down C of L' is SL cone with Gaussian density $\geq 1+\epsilon$.
- In $\mathbb{C}^2$ C is union of planes $\Rightarrow \Theta(y,l) \geq 2 \Rightarrow$ contradiction.

(White) Regularity Theorem $\Rightarrow L$ is smooth.