1. Symplectic geometry review

Let M^{2n} be a manifold with symplectic form $\omega \in \Omega^2(M)$, i.e. satisfying $d\omega = 0$ and $\omega^n \neq 0$. On $\mathbb{R}^{2n} = \mathbb{R}^n_q \times \mathbb{R}^n_p = T^*\mathbb{R}^n$ we have the standard symplectic form

$$\omega_0 = \sum_{k=1}^{n} dq_k \wedge dp_k.$$

By the following theorem, all symplectic geometries are locally standard:

Theorem 1.1 (Darboux's theorem)**.** *For all $x \in (M, \omega)$ there exists a chart (U, ϕ), $x \in U$ such that $\phi(x) = 0 \in \mathbb{R}^n$ and $\phi^*\omega_0 = \omega$.*

So there are no local invariants and all interesting questions are global.

2. Applications of pseudoholomorphic curves

2.1. **Application 1: Exotic spaces.** $n = 1$: Manifold with a volume form ω is symplectic. Symplectomorphisms are area-preserving maps.

$n = 2$: M^4 - lots of cool stuff happens in 4-dimensions: can we have exotic symplectic $\mathbb{R}^4$? Answer: Yes! But...

Theorem 2.1 (Gromov)**.** *Let (M^4, ω) have $\pi_2(M) = 0$. Suppose that*

$$(M^4 \backslash K_M, \omega) \underset{symp}{\cong} (\mathbb{R}^4 \backslash K_{\mathbb{R}^4}, \omega_0)$$

where $K_M \subset\subset M^4$ and $K_{\mathbb{R}^4} \subset\subset \mathbb{R}^4$. Then

$$(M^4, \omega) \underset{symp}{\cong} (\mathbb{R}^4, \omega_0)$$

One way to interpret this is through contact geometry. This can be proven with pseudoholomorphic curves.

2.2. **Application 2: Non-squeezing.** Now suppose we have two symplectic manifolds of the same dimension. Can we embed symplectically?

$$(M_1^{2n}, \omega_1) \underset{\text{symp}}{\hookrightarrow} (M_2^{2n}, \omega_2)$$

Certainly we need to have

$$\int_{M_2} \omega_2^n = \text{Area}(M_2^{2n}) \geq \text{Area}(M_1^{2n}) = \int_{M_1} \omega_2^n$$

but this is not sufficient:

Theorem 2.2 (Gromov non-squeezing)**.** $(B_r^{2n}, \omega_0) \hookrightarrow (B_R^2 \times \mathbb{R}^{2n-2}, \omega_0)$ *if and only if $r \leq R$.*

If you replace "symplectic form" with "volume form" then the volume being smaller is sufficient for an embedding to exist. This was the first time that people realised studying symplectic forms is very different to studying volume forms.

2.3. **Application 3: Foliations of $\mathbb{CP}^2$.** An example of a symplectic 4-fold is

$$(\mathbb{CP}^2, \omega_{FS})$$

Can we characterise this manifold? $\mathbb{CP}^2$ is foliated by embedded S^2s which meet at one point:

- $\omega_{FS}|_{S^2} \neq 0$, i.e. symplectic S^2s.
- $[S^2] \cdot [S^2] = 1$, i.e. self-intersection number is 1.

Theorem 2.3 (Gromov-McDuff)**.** *Let (M^4, ω) be compact, connected and*

- *there exists $C \subset M$ symplectic s.t. $C = S^2$, $\omega|_C \neq 0$, $[C] \cdot [C] = 1$, and*
- *there does not exist symplectically embedded S^2 in M s.t. $[S^2] \cdot [S^2] = -1$.*

Then $(M^4, \omega) \underset{symp}{\cong} (\mathbb{CP}^2, c\omega_{FS})$ for some $c > 0$.

3. Strategy

- Start with one pseudoholomorphic curve C
- Look at moduli space $\mathcal{M}$ of C and show it's nice.
- This allows us to build a family of pseudoholomorphic curves filling out M.

What do we mean by "look at the moduli space"?

- Show $\mathcal{M} \neq \varnothing$.
- Show $\mathcal{M} = \bar{\partial}_J^{-1}(0)/G$, where $\bar{\partial}_J$ is the Cauchy-Riemann operator.
- Local structure: The Fredholm theory for $\bar{\partial}_J$ and transversality tells us what's happening locally.
- Global structure: Compactness theory for $\bar{\partial}_J$ leads to $\bar{\mathcal{M}}$, the compactification of $\mathcal{M}$.

4. Pseudoholomorphic curves

Consider $f : \mathbb{R}^2 \to \mathbb{R}^2$, where $f = (u, v)$ with $u, v : \mathbb{R}^2 \to \mathbb{R}$. Define the matrices

$$j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then we calculate that

$$J \circ df \circ j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -v_y & v_x \\ u_y & -u_x \end{pmatrix}$$

and so if

$$df + J \circ df \circ j = \begin{pmatrix} -v_y + u_x & v_x + u_y \\ u_y + v_x & -u_x + v_y \end{pmatrix} = 0$$

i.e. if $J \circ df = df \circ j$, the Cauchy-Riemann equations are satisfied and we conclude that f is holomorphic. The converse also holds.

Now generalise the range $\mathbb{R}^2$ to $\mathbb{R}^{2n} = \mathbb{C}^n$ and define

$$\bar{\partial}_J f = \tfrac{1}{2}(df + J \circ df \circ j)$$

(where we have extended J to a $2n$-dimensional square matrix in the obvious way.) Then $df + J \circ df \circ j = 0$ if and only if $f : \mathbb{C} \to \mathbb{C}^n$ is a holomorphic curve. So we have a condition for f to be holomorphic.

Now more generally on a symplectic manifold (M, ω) we need to find an almost complex structure, i.e. $J \in \mathrm{End}(TM)$ with $J^2 = -\mathrm{Id}$. We say that J is

- ω-compatible iff $g(\cdot, \cdot) = \omega(\cdot, J\cdot)$ for some Riemannian metric g.
- ω-tame iff $\omega(X, JX) > 0$ for all X. (This is not ω-compatible since it isn't necessarily symmetric, though it could be symmetrised.)

Consider now a map $f : \Sigma \to (M, \omega)$ where (Σ, j) is a complex curve. Denote by $\mathcal{J}$ the set of ω-compatible (or ω-tame) J. Then $\mathcal{J}$ is non-empty and contractible. Choose $J \in \mathcal{J}$, and choose conformal coordinates $s + it$ on (Σ, j). Then

$$\bar{\partial}_J f \overset{\text{loc}}{=} \tfrac{1}{2}(\partial_s f + J\partial_t f)ds + \tfrac{1}{2}(\partial_t f - J\partial_s f)dt.$$

These are the non-linear Cauchy-Riemann equations since J is not fixed but rather dependent on f. Now we say that

$$f : (\Sigma, j) \to (M, \omega, J)$$

is *pseudoholomorphic* (or *J-holomorphic*) if and only if $\bar{\partial}_J f = 0$.

5. Energy of a pseudoholomorphic curve

The energy of f is defined as

$$E(f) = \frac{1}{2} \int_\Sigma |df|^2 \,\mathrm{vol}_\Sigma = \int_\Sigma |\bar{\partial}_J f|^2 \,\mathrm{vol}_\Sigma + \int_\Sigma f^*\omega.$$

The first term is greater than or equal to 0, with equality if and only if f is is pseudoholomorphic. The second term is purely topological. Hence pseudoholomorphic curves minimise energy in their homology class.

6. Moduli space of a pseudoholomorphic curve

Fix $A \in H_2(M)$. The moduli space of a pseudoholomorphic curve is then

$$\mathcal{M} = \{f : (S^2, j) \to (M, \omega, J) \mid \bar{\partial}_J f = 0, f_*[S^2] = A\}/G$$

where $G = \mathrm{PSL}(2, \mathbb{C})$ is the space of Möbius transformations acting on S^2. Note that in general $\mathcal{M}$ is non-compact:

Example 6.1. Let $M = \mathbb{CP}^2$ and, considering $S^2 = \mathbb{CP}^1$, define the map $f_m : \mathbb{CP}^1 \to \mathbb{CP}^2$ by

$$f_m([z_1 : z_2]) = [z_1^2 : z_2^2 : mz_1z_2].$$

Then

$$\begin{aligned} f_m(\mathbb{CP}^1) &= \big\{[z_1 : z_2 : z_3] : z_1z_2 = z_3^2/m^2\big\} \\ &\to \big\{[z_1 : z_2 : z_3] : z_1z_2 = 0\big\} \text{ as } m \to \infty, \end{aligned}$$

which is the union of two $\mathbb{CP}^1$s but is not a map from $\mathbb{CP}^1$.

Now take some Möbius transformation $\phi_m \in G$ such that

$$\phi_m([z_1 : z_2]) = [z_1 : mz_2].$$

Reparametrise by this Möbius transformation:

$$f_m \circ \phi_m([z_1 : z_2]) = f_m([z_1 : mz_2]) = [z_1^2/m^2 : z_2^2 : z_1z_2]$$

Now we find that

$$f_m \circ \phi_m(\mathbb{CP}^1) \to \{[0 : z_1 : z_2]\}$$

which is just a single $\mathbb{CP}^1$! (The same procedure with $\phi_{m^{-1}}$ would give us the other $\mathbb{CP}^1$.)

Theorem 6.2 (Gromov compactness theorem for pseudoholomorphic curves)**.** *Let $J_n \in \mathcal{J}$ (for ω-tame) with $J_n \to J \in \mathcal{J}$ in C^∞. Let $f_n : (S^2, j) \to (M, \omega, J_n)$ be J_n-holomorphic with a uniform bound on the energy, i.e.* $\sup_n(E_n(f_n)) < \infty$. *Then there exists a subsequence of f_n converging to a J-stable map (f, z).*

What is (f, z)? It is a bubble tree. In the above example, we got a pair of spheres. Let us briefly (and without any diagrams since I cannot be bothered to texify them) describe the notation we will use for bubble trees. Consider the bubble tree to be a tree T comprised of vertices α (the centres of each sphere) connected by edges $\alpha E \beta$ (representing the nodal points $z^{\alpha\beta}$ where two spheres with centres α and β touch). Define the subtree $T_{\alpha\beta}$ to be the vertices and edges connected to β which do not contain the edge $\alpha E \beta$.

So (f, z) is a collection

$$(\{f^\alpha\}_{\alpha \in T}, \{z^{\alpha\beta}\}_{\alpha E \beta})$$

where $f^\alpha : (S^2, j) \to (M, \omega, J)$ are J-holomorphic and $z^{\alpha\beta} \in S^2$ are nodal points, such that

(a) $\alpha E \beta \implies f^\alpha(z^{\alpha\beta}) = f^\beta(z^{\beta\alpha})$,
(b) $\alpha \in T \implies z^{\alpha\beta} \neq z^{\alpha\beta'}$ if $\beta \neq \beta'$,
(c) $f^\alpha(S^2) = \{*\} \implies \#Z^\alpha \geq 3$, where $Z^\alpha = \{z^{\alpha\beta} : \alpha E \beta\}$ is the set of nodal points on the α-sphere.

This final condition is stability. Then (f, z) is called a "stable J-holomorphic map of genus 0, modelled on the tree T."

In Sacks-Uhlenbeck, we found an inequality for the energy. Here however, we'll obtain an equality.

Question: What does $f_n \to (f, z)$ really mean? It's *Gromov convergence*. Take $E(f) = \sum_{\alpha \in T} E(f^\alpha)$ and $M_{\alpha\beta}(f) = \sum_{\gamma \in T_{\alpha\beta}} E(f^\gamma)$ (so that $M_{\alpha\beta}$ gives the energy of the subtree $T_{\alpha\beta}$. Then we have the following definition:

Definition 6.3. $f_n \to (f, z)$ means that there exists $\{\phi_n^\alpha\}_{\alpha \in T}, \phi_n^\alpha \in G$ such that

- (Map) For all $\alpha \in T$, $f_n \circ \phi_n^\alpha \xrightarrow{C^\infty} f^\alpha$ on $S^2 \backslash Z^\alpha$. (i.e. up to reparametrisation, $f_n \to f$ away from nodal points.)

- (Energy) There is no energy lost in the limit at a nodal point $z^{\alpha\beta}$:

$$M_{\alpha\beta}(f) = \lim_{\varepsilon \to 0} \lim_{n \to \infty} E\left(f_n^\alpha|_{B_\varepsilon(z^{\alpha\beta})} \right)$$

- (Rescaling) Let $\alpha E \beta$ be an edge. Then

$$(\phi_n^\alpha)^{-1} \circ \phi_n^\beta \xrightarrow{C^\infty} z^{\alpha\beta}$$

on $S^2 \backslash \{z^{\beta\alpha}\}$.

Department of Mathematics, University College London, Gower Street, London, WC1E 6BT, United Kingdom
E-mail address: christopher.evans.13@ucl.ac.uk