Complex curves, twistor spaces and hyperbolic manifolds

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I should say before I begin, to avoid confusion...

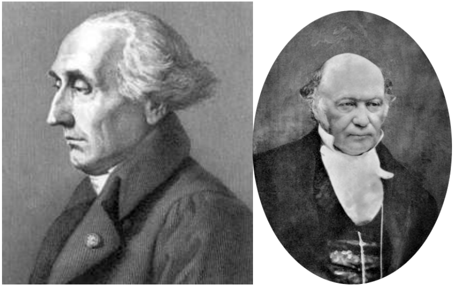
Remark

A complex curve is a two-dimensional object (with a local complex coordinate).

Picture courtesy of Wikipedia.

1770s-1830s

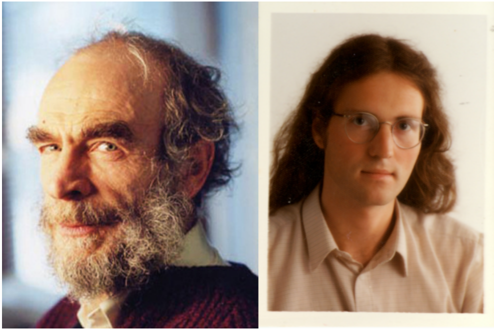
- The ideas of symplectic geometry were born in the works of Lagrange and Hamilton on optics and mechanics.
- Provides a geometrical language for discussing problems in dynamics



Lagrange and Hamilton, courtesy of Wikipedia

1980s:

Gromov, Ruan-Tian, Floer and others introduced some powerful new tools (**quantum cohomology**, **Floer cohomology**).



Gromov and Floer (photos by: Gérard Uferas; Detlef Floer and Michael Link)

- Applications in **dynamics**, **topology**, **enumerative geometry**...
- Tools involve counting complex curves in symplectic manifolds.

Aim

Many powerful tools, too few computations!

- I will explain two of my computations:
  - Theorem 1: Quantum cohomology
  - Theorem 2: Floer cohomology

  for the **twistor spaces of hyperbolic manifolds**.
- The computations are made possible by a beautiful dictionary:

	Hyperbolic Geometry	$\longrightarrow$	Symplectic Geometry	$\longrightarrow$	Algebra
1.	Minimal surfaces	$\longrightarrow$	Complex curves	$\longrightarrow$	Quantum cohomology
2.	Totally geodesic submanifolds	$\longrightarrow$	Lagrangian submanifolds	$\longrightarrow$	Floer cohomology

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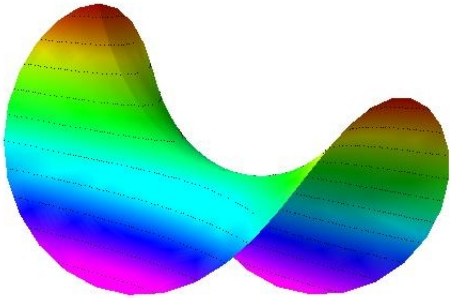
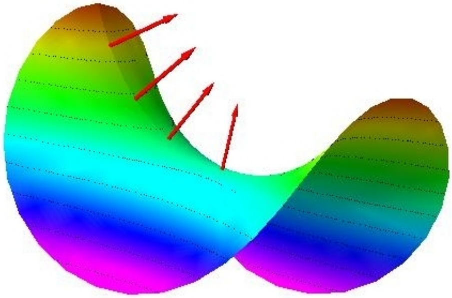
Minimal surfaces to complex curves

- A minimal surface in $\mathbf{R}^3$ is a surface which is stationary for the variation of area.
- A famous example is Enneper's minimal surface, discovered in 1863 by Alfred Enneper:



Minimal surfaces to complex curves

- The link with complex curves comes when we consider the unit normal vectors to the surface.
- These unit normals define a map, the *Gauss map*, from the surface to the unit sphere.



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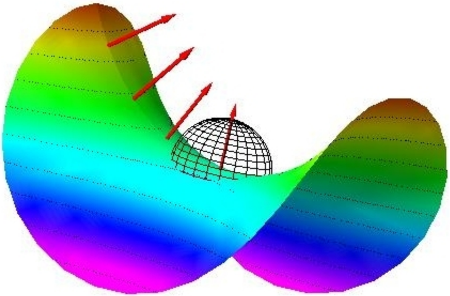

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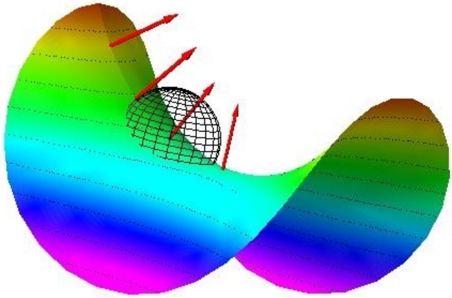

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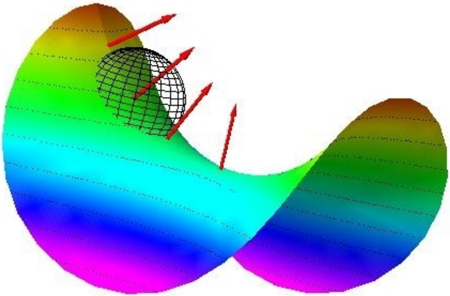

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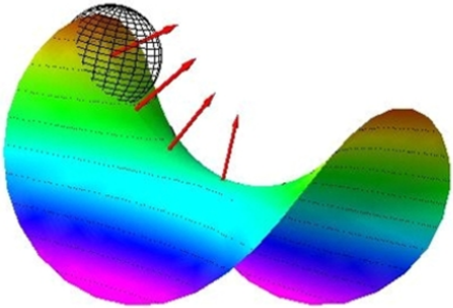
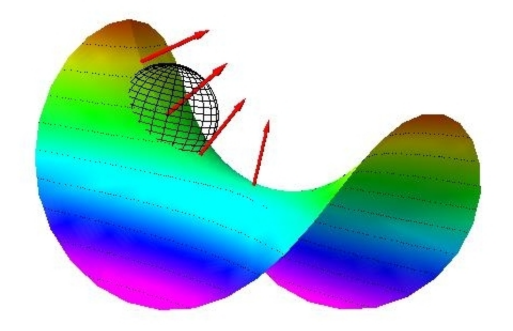


Minimal surfaces to complex curves

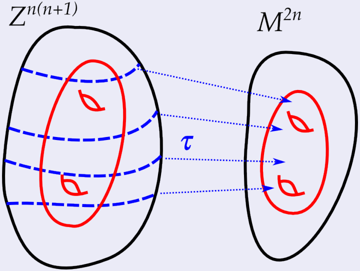
Theorem (Enneper-Weierstrass)

An immersed surface in $\mathbf{R}^3$ is minimal if and only if the Gauss map is holomorphic (i.e. preserves angles).



Minimal surfaces to complex curves

Theorem (Eells-Salamon 1985, Twistor correspondence)



Let M be a Riemannian 2n-manifold. There is a bundle $\tau : Z \to M$, called the **twistor space of** *M, such that:*

- *complex curves in Z project (via τ) to minimal surfaces in M,*
- *moreover any minimal surface arises this way.*

Hyperbolic geometry to symplectic geometry

- We have now explained the first page of the dictionary.

	Hyperbolic Geometry	⟶	Symplectic Geometry	⟶	Algebra
1.	Minimal surfaces	⟶	Complex curves	⟶	Quantum cohomology
2.	Totally geodesic submanifolds	⟶	Lagrangian submanifolds	⟶	Floer cohomology

Hyperbolic geometry to symplectic geometry

- But we haven't explained the headings!

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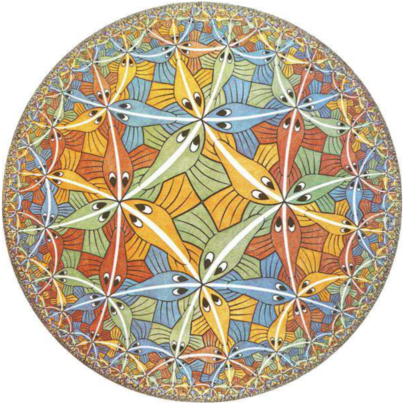
Hyperbolic geometry to symplectic geometry

We are interested in the case when M is **hyperbolic**, i.e. a manifold of constant negative sectional curvature.

In particular:

- a 2-dimensional slice looks locally like this:
- $\mathrm{Vol}(B(r)) \sim e^r$ for small balls.

M. C. Escher, Circle Limit III



Hyperbolic geometry to symplectic geometry

What's nice about hyperbolic M is that:

Theorem (Reznikov 1993)

The twistor space of a hyperbolic manifold is **symplectic***.*

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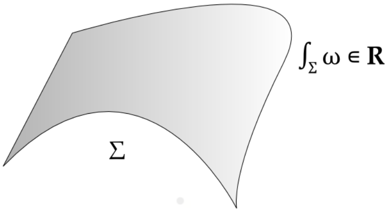
Can integrate ω over surfaces



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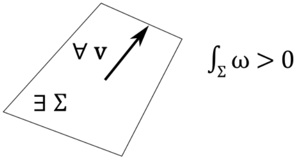

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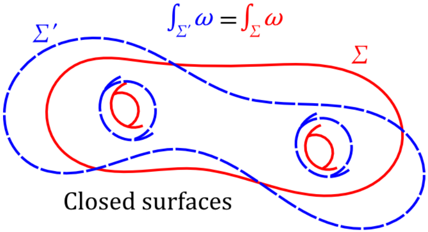

Symplectic geometry to algebra

Let us finish the first line of our dictionary.

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- Sir M. F. Atiyah

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Paraphrasing:

Algebra helps organise complicated geometrical information.

Symplectic geometry to algebra

Theorem (Ruan-Tian, based on ideas of Gromov)

Let X be a symplectic manifold. There is a ring:

- **quantum cohomology ring**, $QH^*(X)$,
- *structure constants encode counts of rational complex curves.*
- $QH^*(X)$ *is* **associative**.

A *rational* complex curve means a holomorphic map from the *Riemann sphere* into X.

Symplectic geometry to algebra

Example (Kontsevich)

Associativity of $QH^*(X) \Rightarrow$ a recursion formula for N_d

$$N_d = \#\ \textit{rational curves of degree } d \textit{ through } 3d-1 \textit{ points in plane}$$

- *19th century geometers struggled to compute these numbers in low degrees*

$$1, 1, 12, 620, 87304, 26312976, \ldots$$

- *Now we have a mindless algorithm for generating as many as we like.*

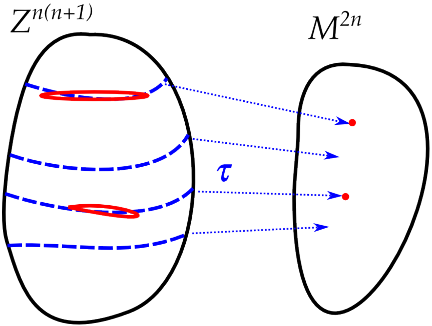
Fact

Many other algebraic symplectic invariants
are controlled by QH^*, e.g. modules over it.

Quantum cohomology of twistor spaces

Key Fact

- *There are no minimal 2-spheres in hyperbolic manifolds!*
- *Hence all rational complex curves live in the fibres of the map* $\tau\colon Z \to M$.



Quantum cohomology of twistor spaces

Key Fact

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Theorem 1 (E. 2011)

The quantum cohomology ring of the twistor space Z of a hyperbolic 6-manifold M is

$$QH^*(Z) = H^*(M; \mathbf{C}[q^{\pm 1}])[c_1]/(c_1^4 = 8c_1\tau^*\chi + 8qc_1^2 - 16q^2)$$

where

- $c_1 \in H^2(Z; \mathbf{C})$ is the first Chern class, χ is the Euler class of M
- q is a formal variable encoding the areas of complex curves.

Quantum cohomology of twistor spaces

Ideas for proof

The proof uses *virtual perturbation theory*.

- This involves computing the Euler class of a bundle over the space of complex curves.
- The bundle is constructed from solutions to an elliptic PDE.
- The Euler class can be computed using the Borel-Hirzebruch formula for fibre integrals.

Totally geodesic submanifolds to Lagrangians

- Now for the second line of our dictionary.

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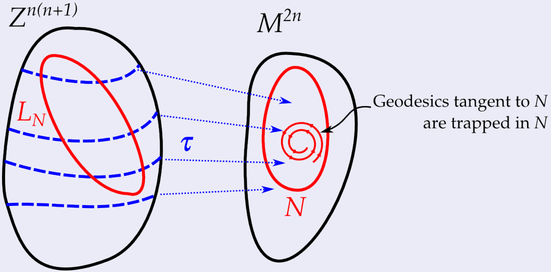
Totally geodesic submanifolds to Lagrangians

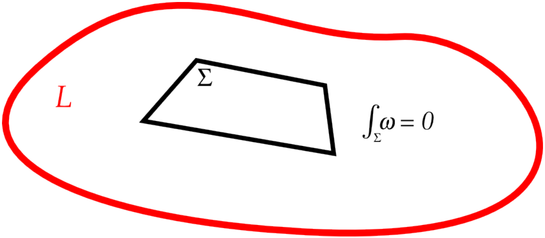
Theorem (Reznikov 1993)



Let M be a 2n-dimensional hyperbolic manifold and let Z be its twistor space. A totally geodesic n-dimensional submanifold $N \subset M$ lifts to a **Lagrangian submanifold** $L_N \subset Z$.

Totally geodesic submanifolds to Lagrangians

- A **Lagrangian submanifold** $L \subset Z$ is characterised by the properties that
  - the integral of ω over any small 2-d rectangle in L vanishes and
  - it has dimension equal to half the dimension of Z.



- For example, complex projective varieties are symplectic, real projective varieties are Lagrangian.
- Lagrangian submanifolds are central in symplectic geometry.

Lagrangians to Floer cohomology

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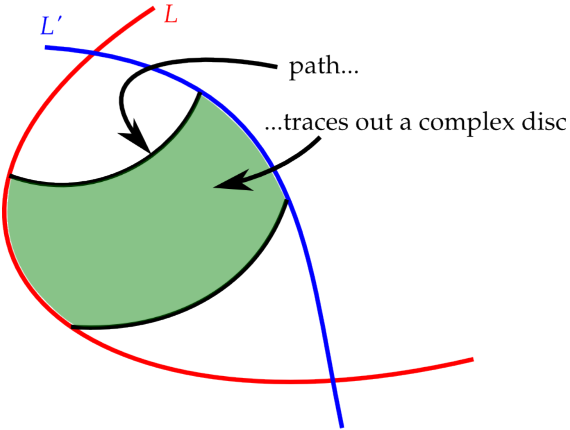
- The Floer cohomology $HF(L)$ is an invariant associated to a Lagrangian submanifold.
- It is a module over quantum cohomology!

$$QH^*(Z) = H^*(M; \mathbf{C}[q^{\pm 1}])[c_1]/(c_1^4 = 8c_1\tau^*\chi + 8qc_1^2 - 16q^2)$$

- c_1 is invertible in $QH^*(Z) \Rightarrow$ restricts how Lagrangians can embed.

Lagrangians to Floer cohomology

- Dimension of $HF(L)$ "counts" the number of intersection points of L with L', a deformation of L.
- The actual definition involves Morse theory on the ∞-dimensional space of paths from L to L'.



Floer cohomology of Reznikov Lagrangians

- Eells-Salamon theorem relates complex discs to minimal surfaces.
- A hopeless ∞-dimensional problem is reduced to something much more manageable.

Theorem 2 (E. 2011)

Let

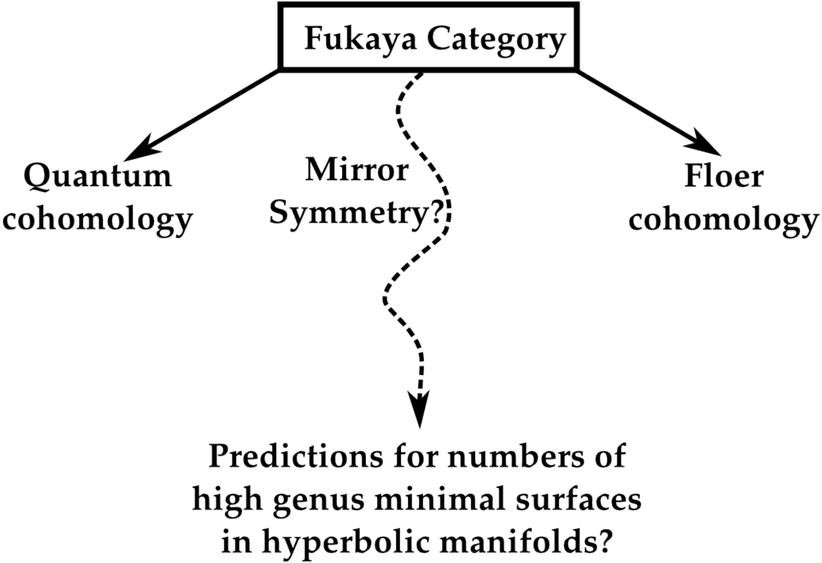
- M be a hyperbolic 6-manifold,
- $N \subset M$ be a totally geodesic 3-manifold.
- L_N be the associated Reznikov Lagrangian.

Then

$$HF^*(L_N) \cong H^*(L_N; \mathbf{C}[t, t^{-1}])$$

where t is a formal variable encoding areas of complex discs.

Bigger picture



Summary

Theorem 1 (E. 2011)

$$QH^*(Z) = H^*(M; \mathbf{C}[q^{\pm 1}])[c_1]/(c_1^4 = 8c_1\tau^*\chi + 8qc_1^2 - 16q^2)$$

$$\text{Minimal surfaces} \xrightarrow{\substack{\text{Eells-Salamon} \\ \text{twistor correspondence}}} \text{Complex curves} \xrightarrow{\text{Ruan-Tian}} \text{Quantum cohomology}$$

Theorem 2 (E. 2011)

$$HF^*(L_N) \cong H^*(L_N; \mathbf{C}[t, t^{-1}])$$

$$\text{Totally geodesic submanifolds} \xrightarrow{\substack{\text{Reznikov} \\ \text{construction}}} \text{Lagrangian submanifolds} \xrightarrow{\text{Floer}} \text{Floer cohomology}$$