Unlinking and unknottedness of monotone Lagrangian submanifolds

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Main result

Joint work with Georgios Dimitroglou Rizell.

Theorem (Unknottedness A)

Let $n \geq 5$ be odd. Suppose $\iota_1, \iota_2 \colon L \to \mathbb{C}^n$ are two monotone Lagrangian embeddings of a torus into the standard symplectic vector space. Then (after possibly reparametrising) there is an isotopy of smooth (not necessarily Lagrangian) embeddings connecting ι_1 and ι_2.

By contrast there are knotted smooth tori, even knotted totally real tori.

Definitions

- Totally real means that $iT_pL \cap T_pL = \{0\}$ for all $p \in L$.
- Lagrangian means $\omega|_{T_pL} = 0$ for all $p \in L$ and T_pL is n-dimensional.
- Lagrangian implies totally real because $iT_pL \perp T_pL$ when L is Lagrangian.
  - To see this, note that if $v \in T_pL$ and $w \in iT_pL$ then
    $$g(v, w) = \omega(v, iw) = 0$$
    because v and iw are in T_pL.

H-principles

There is an *h-principle* for totally real embeddings and for Lagrangian immersions. Note that a totally real embedding (or Lagrangian immersion) ι gives a trivialisation of the complexified tangent bundle

$$TL \otimes \mathbb{C} \xrightarrow{\cong} L \times \mathbb{C}^n$$

which sends $(x, v + iw) \in T_x L$ to $(\iota(x), \iota_* v + i\iota_* w)$.

Theorem (H-principle (Gromov, Lees))

Isotopy classes of totally real embedding (respectively regular homotopy classes of Lagrangian immersion) correspond 1-1 with homotopy classes of trivialisation

$$TL \otimes \mathbb{C} \xrightarrow{\cong} L \times \mathbb{C}^n$$

together with a smooth isotopy class of embeddings (respectively immersions) $L \to \mathbb{C}^n$.

Rigidity phenomena

By contrast there is no h-principle for Lagrangian embeddings so various kinds of *rigidity phenomena* can occur:

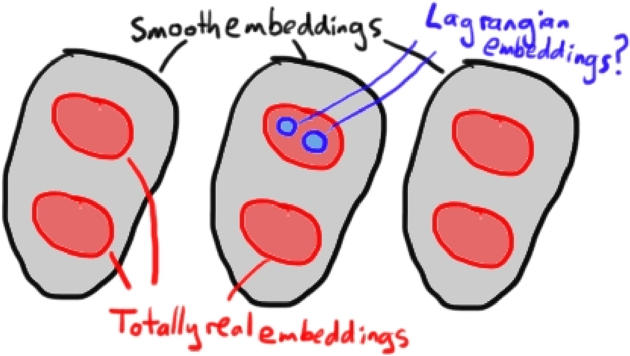


Figure : (a) There may be many isotopy classes of totally real embeddings, only a few of which contain Lagrangian embeddings. (b) There may be several isotopy classes of Lagrangian embeddings which are isotopic as totally real submanifolds. Theorem A above concerns rigidity of type (a).

More rigidity phenomena

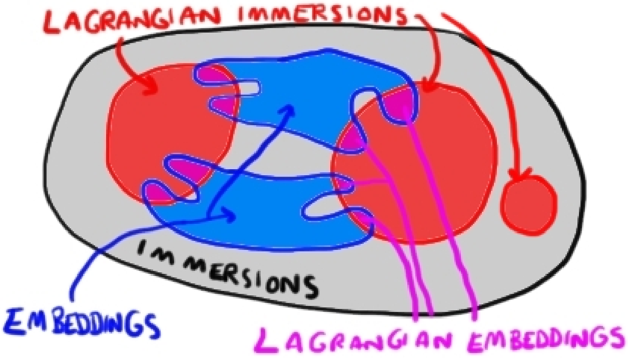
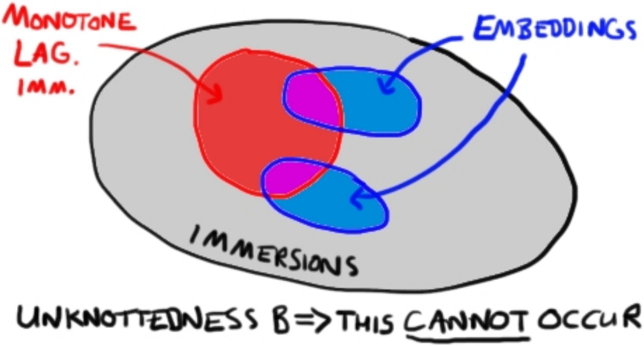


Figure : Amongst all immersions there may be (a) homotopy classes of Lagrangian immersion containing no Lagrangian embedding, (b) isotopy classes of Lagrangian embedding which are isotopic as Lagrangian immersions but not as smooth embeddings. Our second theorem will rule out (b) for monotone Lagrangians.

Theorem (Unknottedness B)

Suppose $n \geq 4$. Two monotone Lagrangian embeddings of a torus in $\mathbb{C}^n$ are smoothly isotopic if they are homotopic as Lagrangian immersions.



Summary

Theorem (Unknottedness Theorems)

- *Let $n \geq 5$ be odd. Suppose $\iota_1, \iota_2 \colon L \to \mathbb{C}^n$ are two monotone Lagrangian embeddings of a torus into the standard symplectic vector space. Then (after possibly reparametrising) there is an isotopy of smooth (not necessarily Lagrangian) embeddings connecting ι_1 and ι_2.*
- *Suppose $n \geq 4$. Two monotone Lagrangian embeddings of a torus in $\mathbb{C}^n$ are smoothly isotopic if they are isotopic as Lagrangian immersions.*

To prove these theorems, we must ask:

Question

How does one prove two n-dimensional submanifolds of $\mathbb{C}^n$ are smoothly isotopic?

Haefliger-Hirsch theory

Question

How does one prove two n-dimensional submanifolds of $\mathbb{C}^n$ are smoothly isotopic?

The answer was given by Haefliger and Hirsch in 1963

Theorem (Haefliger-Hirsch)

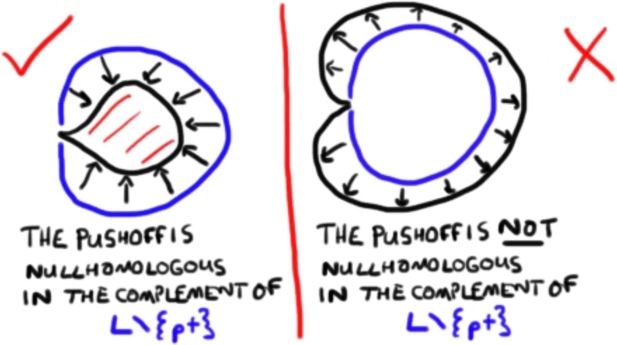
Let $n \geq 4$ and let L be a closed orientable n-manifold. Smooth isotopy classes of embedding $L \to \mathbb{C}^n$ correspond (noncanonically) 1-1 with elements of

$$H_1(L; \mathbb{Z}) \text{ when } n \text{ is odd}, \; H_1(L; \mathbb{Z}/2) \text{ when } n \text{ is even.}$$

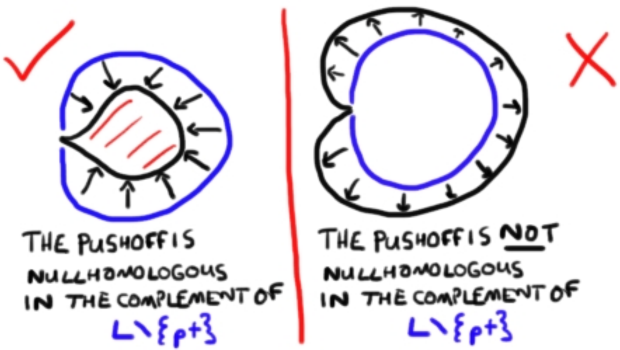
Haefliger-Hirsch theory

More precisely,

- To any embedding ι there is a normal vector field (the *Haefliger-Hirsch field*) η on $L \setminus \{\mathrm{pt}\}$ such that pushing $\iota(L)$ off itself using η gives $\iota_\eta(L)$ which is nullhomologous in $\mathbb{C}^n \setminus (L \setminus \{\mathrm{pt}\})$.



- The obstruction to constructing a smooth isotopy between embeddings ι_1 and ι_2 is the obstruction to finding a homotopy of normal vector fields over the $n-1$-skeleton.



Self-linking

So Haefliger-Hirsch theory reduces smooth isotopy problems to a study of "self-linking".

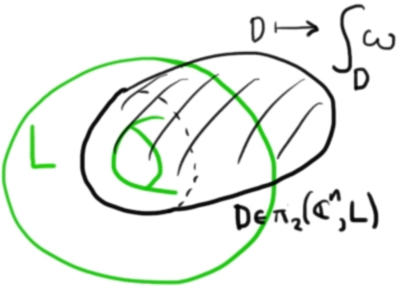
Definition (Linking of n-dimensional submanifolds)

$L \subset X$ links $M \subset X$ if $[L] \neq 0 \in H_n(X \setminus M; \mathbb{Z})$.

Area and linking

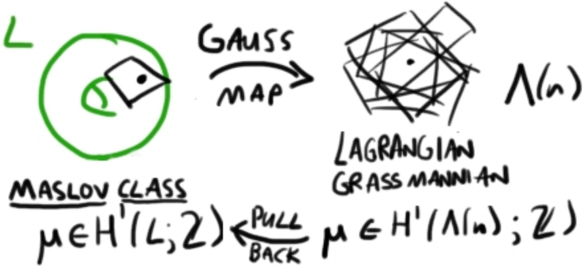
- Clearly for circles in $\mathbb{C}$, if area$(M) \geq$ area(L) then L does not link M.
- What is the "area" of a Lagrangian submanifold?
- There is a homomorphism $\int \omega \colon \pi_2(\mathbb{C}^n, L) \to \mathbb{R}$ defined by integrating ω over a disc.
- Since $\mathbb{C}^n$ is contractible, the homotopy long exact sequence of $L \subset \mathbb{C}^n$ implies

$$\pi_2(\mathbb{C}^n, L) \cong \pi_1(L)$$



Maslov class

- There is another homomorphism $\mu\colon \pi_1(L) \to \mathbb{Z}$ defined as follows:



Definition

A Lagrangian is monotone *if, for some* $K > 0$,

$$\int_A \omega = K\mu(A)$$

We restrict attention to monotone Lagrangians because

- the ω-area homomorphism is particularly simple for them and
- more importantly, Floer theory is very much easier in the monotone setting (Oh, Biran-Cornea).

Are we cheating?

We want to prove a rigidity theorem. We cannot say anything about Lagrangian embeddings in general, only the monotone ones, so are we cheating?

No

There is an h-principle for monotone Lagrangian immersions, so our theorem is a rigidity result for that h-principle.

Theorem (Unlinking)

Suppose $\iota_1, \iota_2 \colon T^n \to \mathbb{C}^n$ *are monotone Lagrangian embeddings with*[a] $K_1 \leq K_2$. *Then* $\iota_1(T) = L_1$ *does not link* $\iota_2(T^n) = L_2$.

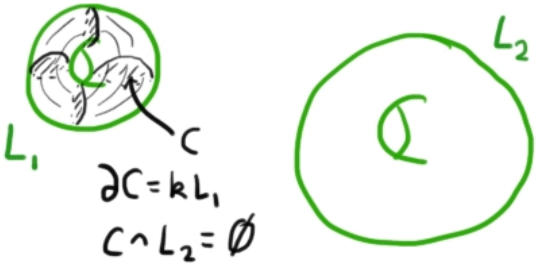
[a] K_i is the monotonicity constant of ι_i.

This is the key result which lets us deduce our unknottedness results via Haefliger-Hirsch theory.

Proof of unlinking theorem I

- The idea is the use J-holomorphic discs with boundary on L_1 to construct a chain C with $\partial C = kL_1$ for some $k \neq 0 \in \mathbb{Z}$.
- We need to pick J such that these discs are disjoint from L_2.
- This will show $k[L_1] = 0 \in H_n(\mathbb{C}^n \setminus L_2; \mathbb{Z}) \cong H^{n-1}(L; \mathbb{Z}) = \mathbb{Z}^n$ and hence

$$[L_1] = 0 \in H_n(\mathbb{C}^n \setminus L_2; \mathbb{Z}).$$



Step 1: Finding holomorphic discs

Theorem (Buhovksy)

If $L \subset \mathbb{C}^n$ is a monotone Lagrangian torus then there exists a Maslov 2 homology class[a] $\beta \in H_1(L;\mathbb{Z})$ *such that for generic J,*

$$\mathrm{ev}\colon \mathcal{M}_{0,1}(L,J;\beta) \to L$$

has non-zero degree.

[a] i.e. $\mu(\beta) = 2$.

Here $\mathcal{M}_{0,1}(L,J;\beta)$ is the moduli space of pairs (u,z) where

- $u\colon D^2 \to \mathbb{C}^n$, $\partial u\colon S^1 \to L$
- $[\partial u] = \beta$ and u is J-holomorphic,
- $z \in \partial D^2$,

divided by the action of $\mathbb{P}SL(2,\mathbb{R})$, where $\phi \in \mathbb{P}SL(2,\mathbb{R})$ acts by

$$(u,z) \mapsto (u \circ \phi^{-1}, \phi(z))$$

Step I: Finding holomorphic discs

The map $\mathrm{ev}\colon \mathcal{M}_{0,1}(L, J; \beta) \to L$ is the *evaluation map*:

$$\mathrm{ev}(u, z) = u(z)$$

If there are k marked points then the moduli space (generically) has dimension

$$n + \mu(\beta) + k - 3$$

in our case

$$n + 3 - 3 = n$$

Moreover, the Maslov 2 discs form a compact moduli space by monotonicity, so the evaluation map has a degree.

Aside: Buhovsky's theorem

How do we find such discs?

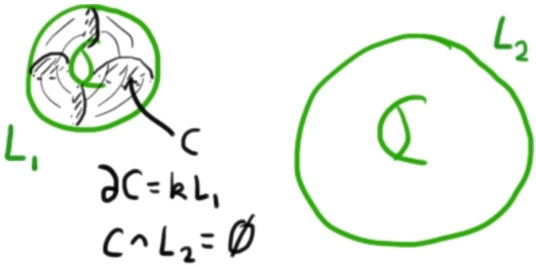
- There is an invariant of monotone Lagrangian submanifolds called *Floer homology*.
- This invariant is a ring $\mathrm{HF}(L)$.
- If L can be displaced from itself (e.g. in $\mathbb{C}^n$ by translation) then

$$\mathrm{HF}(L) = 0.$$

- There is a spectral sequence starting from $H_*(L; \mathbb{Z}[q, q^{-1}])$ converging to $\mathrm{HF}(L)$.
- The differentials in this sequence are defined as degrees of evaluation maps on moduli spaces of holomorphic discs, so $\mathrm{HF}(L) = 0$ means many differentials are nonzero, hence many discs.
- By considering the product structure on the ring one can actually deduce that the spectral sequence collapses at E_2, and this is the differential which counts Maslov 2 discs.

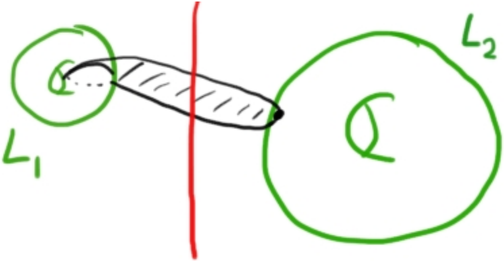
Reminder

We want to show that, for some J, these discs are disjoint from L_2. We'll use neck-stretching.



Step 2: Stretching the neck

Assume (for contradiction) that for all J there is a J-disc with boundary on L_1 representing the class β which intersects L_2. Stretch the neck around L_2.



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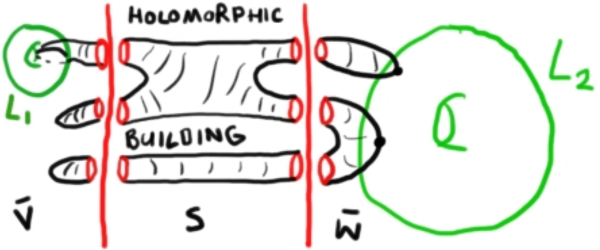
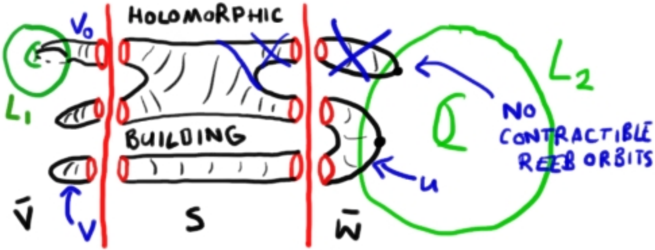
Step 2: Stretching the neck

In the limit our discs become holomorphic buildings with asymptotic Reeb orbits.



- Because the disc always hits L_2, the limit building has a component u in the Weinstein neigbourhood $\bar{W}$.
- The boundary of the Weinstein neighbourhood W is the unit sphere bundle of the cotangent bundle of L_2 and the Reeb flow is the geodesic flow on the torus.

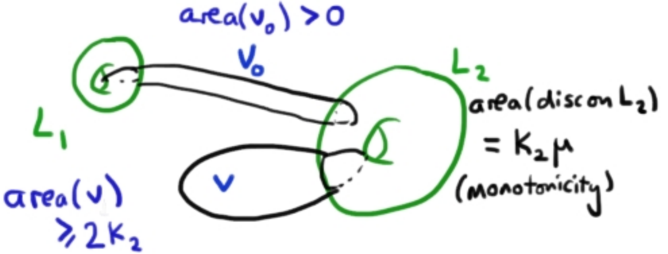
- The torus has no contractible geodesics, so the domain of u has at least two punctures.
- In particular, one of the punctures of u must be capped off[1] by a disc v in $\bar{V} = \mathbb{C}^n \setminus L_2$.
- There is also a component v_0 containing the boundary of the holomorphic building on L_1.



[1]It cannot be capped off by a holomorphic disc in the symplectisation by a maximum principle.

Step 2: Stretching the neck

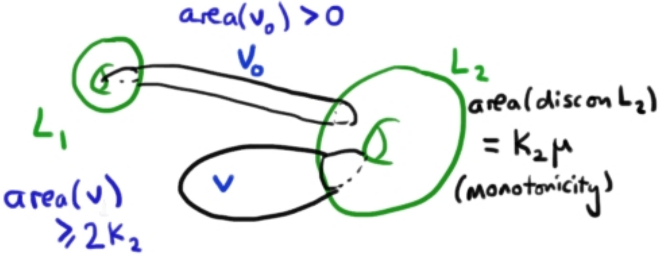
- Now we can interpret $v\colon D^2 \to \mathbb{C}^n \setminus L_2$ as a holomorphic disc with boundary on L_2.
- In particular it represents an element of $\pi_2(\mathbb{C}^n, L_2)$ with positive area (at least $2K_2$ by monotonicity).



- But the minimal positive area of a disc on L_2 is $2K_2$.
- The sum of the areas of all components of the limit building is equal to the area of the original (Maslov 2) disc which was $2K_1$. Since these areas are all positive by holomorphicity

$$2K_1 \geq \text{area}(v) + \text{area}(v_0) > 2K_2$$

which contradicts the assumption $K_1 < K_2$.



Self-linking corollary

Our unknottedness results follow from the following:

Corollary

Let $\iota\colon T^n \to \mathbb{C}^n$ be a monotone Lagrangian torus. Let $\sigma\colon T^n \to S^1$ be a circle-valued function such that $[d\sigma/2\pi] = \mu$ and let $\nabla\sigma$ be its gradient. Then $V = J\nabla\sigma$ is the Haefliger-Hirsch field of ι.

Proof.

The small pushoff of $\iota(T^n)$ along V is a monotone Lagrangian torus with smaller monotonicity constant, hence is nullhomologous in the complement of $\iota(T^n)$. $\square$

- We don't need L to be diffeomorphic to a torus. For the unlinking theorem we just need
  - A non-zero degree of evaluation map for L_1,
  - For instance, we could take an orientable, spin manifold L_1 such that the universal cover has no odd-degree cohomology (discs come from Damian's lifted Floer homology).
  - A metric with no contractible geodesics on L_2.
  - L_1 orientable (for a fundamental class) and $H_1(L_2;\mathbb{Z})$ torsionfree (otherwise we only deduce that $[L_1]$ is torsion in $H_n(\mathbb{C}^n \setminus L_2;\mathbb{Z})$).

  and L_1 need not be diffeomorphic to L_2.

- For Unknottedness B (isotopic as Lagrangian immersions means isotopic through smooth embeddings) we just need L
  - to be spin and orientable,
  - to have a metric with no contractible geodesics[2],
  - to have $H_1(L;\mathbb{Z})$ torsionfree and
  - to admit a circle-valued function $\sigma\colon L \to S^1$ with no critical points such that $[d\sigma/2\pi] = \mu$.

[2] which implies that the universal cover is contractible.

Conclusion

We have proved unlinking and unknottedness results for certain monotone Lagrangian submanifolds in $\mathbb{C}^n$. The natural open question is:

- Can one prove an unlinking result for general Lagrangian embeddings?
  - There may exist Maslov zero Lagrangians in $\mathbb{C}^n$ - these would not have as many holomorphic discs as we need to run our argument...
- ...so can one find counterexamples?
  - Possibly, using forthcoming work of Eliashberg-Ekholm-Murphy-Smith on flexibility for loose Legendrian submanifolds?

Thank you for listening.