Symplectic topology and algebraic geometry II: Lagrangian submanifolds

Jonathan Evans

UCL

26th January 2013

Addendum to last talk:

- People pointed out last time that the argument we gave was very special to $\mathbb{D}_5$.
- This is absolutely true and I used it as a purely pedagogical way to introduce holomorphic curves.
- Seidel gave it for similar reasons in Example 1.13 his "Lectures on 4-dimensional Dehn twists".
- He also used Floer homology, his exact triangle and the structure of quantum cohomology to prove (loc. cit. Theorem 0.5):

Theorem (Seidel)

Let X be a complete intersection surface other than $\mathbb{CP}^2$ or the quadric. The squared Dehn twist is nontrivial in the symplectic mapping class group.

You should read his lectures. They're great.

Lagrangian submanifolds

Remember that a symplectic form is *nondegenerate*, i.e. the biggest isotropic subspaces of tangent spaces are n-dimensional.

Definition

- An n-dimensional isotropic subspace is called a Lagrangian subspace.
- A submanifold whose tangent spaces are Lagrangian is called a Lagrangian submanifold.

Lagrangians in projective varieties

Lagrangian submanifolds arise in complex projective geometry in two ways:

- As real loci of complex varieties (fixed point sets of an antisymplectic involution).
- As vanishing cycles of nodal degenerations.

We will look at these in this order. The plan is:

- prove that a smooth real Fano 3-fold cannot be a hyperbolic 3-manifold,
- briefly discuss the classification of Lagrangian spheres (vanishing cycles) in certain algebraic surfaces.

Real projective varieties

- Complex conjugation $c\colon \mathbb{CP}^N \to \mathbb{CP}^N$ is an antisymplectic involution.
- In other words, $c^2 = 1$ and $c^*\Omega = -\Omega$ where Ω is the Fubini-Study form.
- The fixed locus $\mathbb{RP}^N$ is a Lagrangian submanifold because $\Omega|_{\mathbb{RP}^N} = -\Omega|_{\mathbb{RP}^N}$.
- If a projective variety is cut out by real polynomials then it is preserved by c and hence its intersection with $\mathbb{RP}^N$ is a Lagrangian submanifold.

Kollar's work

- Kollar developed the minimal model program over $\mathbb{R}$ in dimension 3.
- In particular he gave restrictions on the topology of a real projective variety birational to $\mathbb{P}^3$.
- However, to rule out certain cases (like hyperbolic or sol 3-manifolds) one needs techniques from symplectic geometry.
- These arguments were provided by Viterbo-Eliashberg and Welschinger-Mangolte respectively.

We will prove:

Theorem (Viterbo-Eliashberg)

A real Fano 3-fold cannot be a hyperbolic 3-manifold.

Remarks

- The Viterbo-Eliashberg theorem uses a technique called *neck-stretching* to construct an almost complex structure with desired properties.
- This idea originated in gauge theory and Riemannian geometry and first came into symplectic geometry in the work of Hofer.
- I will explain the idea of neck-stretching around a Lagrangian submanifold.
- I will also use neck-stretching in my third talk.
- I hope this excuses a long technical section...

Stretching the neck I

Theorem (Weinstein neighbourhood theorem)

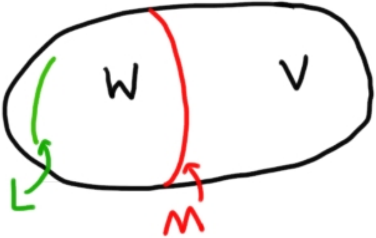
Given a Lagrangian submanifold $L \subset X$ of a symplectic manifold there is a neighbourhood $W \supset L$ which is symplectomorphic to a neighbourhood of the zero-section in T^*L.

Some words of explanation:

- T^*L is the cotangent bundle.
- A point $(x, \eta) \in T^*_x L$ is a 1-form η on the tangent space T_L.
- Given a vector v at $(x, \eta) \in T^*L$ you can project v to L and apply η to get a number.
- The result is called $\lambda(v)$.
- λ is the canonical 1-form on T^*L.
- $-d\lambda$ is a symplectic form.

Stretching the neck II

- Let W be a Weinstein neighbourhood of L.
- Let $M = \partial W$.
- Let $V = X \setminus W$.



Stretching the neck III

M inherits some extra structure, namely a 1-form α pulled back from λ.

Definition (Contact distribution)

- *α is a contact form, i.e.*

$$\alpha \wedge (d\alpha)^{n-1} \textit{ is a volume form on } M.$$

- *$\xi = \ker \alpha$ is a $2n-2$-plane field called the contact distribution.*
- *There is a vector field R_α transverse to ξ called the Reeb field such that*

$$\alpha(R_\alpha) = 1 \textit{ and } d\alpha(R_\alpha, \cdot) = 0.$$

- *Near M there is a vector field called the Liouville vector field v which is transverse to M. This is the outward radial field in the cotangent bundle.*

Stretching the neck IV

Use the flow of the Liouville field v to find a neighbourhood N of M, identified with $M \times (-\epsilon, \epsilon)$. Pick a $J_0 \in \mathcal{J}_\omega$ which is:

- invariant under the flow of v,
- preserves ξ,
- satisfies $J_0 v = R_\alpha$.

We call such an almost complex structure *adapted to L*.

An adapted almost complex structure

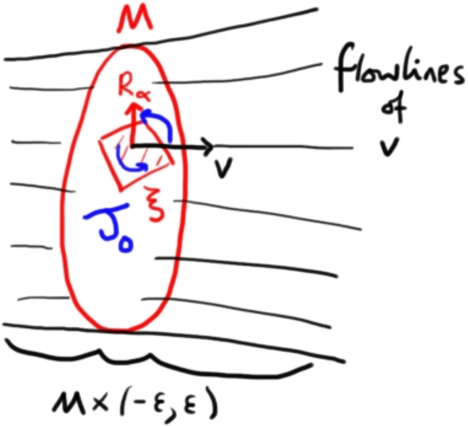


Figure : Our invariant almost complex structure J_0 preserves the contact distribution ξ on M and sends v (the Liouville field) to R_α (the Reeb field).

Stretching the neck V

Neck-stretching takes an almost complex structure J_0 adapted to L and produces a 1-parameter family of degenerating almost complex structures $\{J_T\}_{T\in[0,\infty)}$. Note that $X = W \cup N \cup V$ where $N \cong M \times (-\epsilon, \epsilon)$ and J_0 is extended arbitrarily over W and V.

- Since J_0 is invariant on N under the Liouville flow, one can replace N by $N_T = (-T-\epsilon, T+\epsilon)$.
- Define

$$X_T = W \cup N_T \cup V$$

  and J_T to be the almost complex structure which agrees with J_0 on W and V and is invariant on N_T.

Definition

The family J_T is called the neck-stretching sequence for the adapted almost complex structure J_0.

The key to understanding J_T-holomorphic curves for large T is the *symplectic field theory compactness theorem*:

Theorem (SFT compactness)

If $\{u_T : \mathbb{CP}^1 \to X_T\}_{T\in[0,\infty)}$ is a sequence of J_T-holomorphic curves (in a given homology class) then there is a subsequence u_{T_i}, $T_i \to \infty$, and a sequence of reparametrisations ϕ_i such that $u_i = u_{T_i} \circ \phi_i$ converges to a holomorphic building *in X_∞.*

- I won't talk about the nature of the convergence (Gromov-Hofer convergence).
- X_T breaks up into noncompact pieces as $T \to \infty$:
  - $\bar{W}$, the cotangent bundle of L
  - $S = M \times (-\infty, \infty)$,
  - $\bar{V}$, the complement $X \setminus L$. and

$$X_\infty = \bar{W} \cup S \cup \bar{V}.$$

Holomorphic buildings I

Holomorphic buildings are certain noncompact holomorphic curves. Before defining them, let me give you an example of the kind of thing which is allowed.

Example

On the subset $S = M \times \mathbb{R}$ we have $J_\infty v = R_\alpha$. Therefore a closed orbit of the Reeb vector field R_α will trace out a holomorphic cylinder under the flow of the Liouville vector field v.

- The ends of our noncompact manifolds look like $S \times [0, \infty)$ or $S \times (-\infty, 0]$.
- Holomorphic buildings satisfy a condition called "finiteness of energy" which ensures that they are asymptotic to holomorphic cylinders on closed Reeb orbits.

Holomorphic buildings II

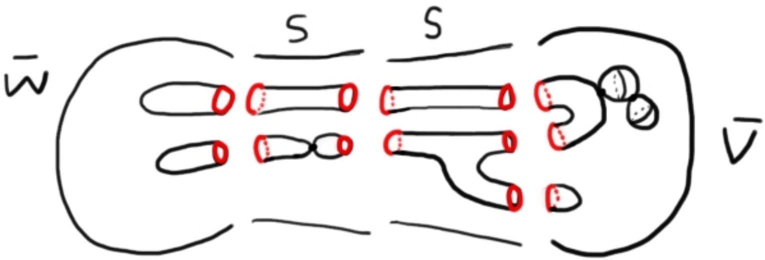


Figure : A holomorphic building in X_∞, asymptotic Reeb orbits drawn in red. These orbits must match up.

Now I want to use these ideas to prove the Viterbo-Eliashberg theorem:

Theorem (Viterbo-Eliashberg)

A real Fano 3-fold cannot be a hyperbolic 3-manifold.

Proof of Viterbo-Eliashberg theorem I

Suppose that L is a hyperbolic 3-manifold embedded as a Lagrangian submanifold of a Fano 3-fold.

Step 1: Hyperbolic geometry to Reeb dynamics

- Recall that the contact hypersurface M essentially the unit cotangent bundle of L.
- In fact, the Reeb vector field is the (co)geodesic vector field and the Reeb orbits are the geodesics.
- On a hyperbolic manifold there is a unique closed geodesic in every noncontractible free homotopy class of loops.

Proof of Viterbo-Eliashberg theorem II

Step 2: Fano gives holomorphic curves

- By Mori theory, a Fano 3-fold is uniruled (i.e. there is a homology class A such that through every point there is a holomorphic sphere homologous to A).
- Kollar-Ruan proved that these uniruling curves persist as J-holomorphic spheres for arbitrary compatible J.
- In particular, one can take a neck-stretching sequence J_t for L.
- For each point $p \in L$ there is a uniruling J_t-sphere through p.
- Using the SFT compactness, you extract a punctured holomorphic curve in T^*L passing through every point of L.

Proof of Viterbo-Eliashberg theorem III

To summarise, in Step 2 we found punctured holomorphic curves passing through every point of our Lagrangian L. By Step 1, these are asymptotic to closed geodesics in the unit cotangent bundle of a hyperbolic manifold. The punchline is the following:

Proposition (Holomorphic curves and Reeb dynamics)

Let L be a hyperbolic n-manifold, $n \geq 3$. For generic J, the moduli space of (simple) punctured holomorphic curves in T^*L is discrete (0-dimensional).

In particular, only a countable set of points in L can be hit by the limits of the uniruling curves from Step 2, a contradiction.

Proof of Viterbo-Eliashberg theorem IV

Proof of Proposition.

Without mentioning various transversality results about simple punctured holomorphic curves, all we need to prove is that the expected dimension of the moduli space is zero. The formula for the expected dimension of a moduli space of punctured holomorphic curves in a cotangent bundle where the geodesic flow is nondegenerate like this one is:

$$(n-3)(2-s)+\sum_{i=1}^{s}\mathrm{ind}(\gamma_i)$$

where the s punctures are asymptotic to geodesics γ_i and $\mathrm{ind}(\gamma_i)$ denotes the Morse index. ...

Recall

Closed geodesics are critical points of the energy functional on loopspace. The Morse index is the dimension of the negative eigenspaces of the Hessian, i.e. the number of linearly independent directions in loopspace in which the energy can decrease. Geodesics in hyperbolic manifolds are action-minimisers, so $\mathrm{ind}(\gamma_i) = 0$. Moreover, they are noncontractible.

Proof of Proposition (continued).

- Noncontractibility implies $s \geq 2$. Otherwise the punctured curve would project to L and give a nullhomotopy of its asymptotic orbit.
- The condition $n \geq 3$ implies that $(n-3)(2-s) \leq 0$.
- Therefore the expected dimension

$$(n-3)(2-s) + \sum_{i=1}^{s} \mathrm{ind}(\gamma_i)$$

is at most zero. □

This completes the proof that there is no hyperbolic Lagrangian submanifold in a Fano manifold.

- The two hypotheses (hyperbolic versus Fano) are very different in flavour (Riemannian geometric versus algebro-geometric).
- The technology of punctured holomorphic curves and neck-stretching allows us to translate between these two worlds and prove a theorem.

Observe...

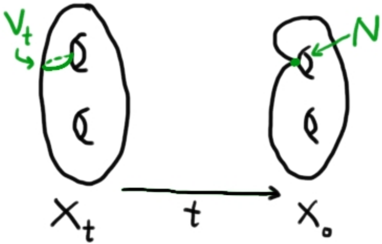
...while most of our theorems have been applications of ideas from algebraic geometry to symplectic problems, this theorem is a result in (real) algebraic geometry which seems to need a symplectic proof.

Vanishing cycles

Symplectic parallel transport

Recall from yesterday's lecture that if (X_t, ω_t) is a 1-parameter family of smooth complex projective varieties then we can define a symplectic parallel transport map $\phi_{s,t}\colon X_s \to X_t$ such that $\phi_{s,t}^*\omega_t = \omega_s$.

Now suppose that X_0 is actually a variety with a node N.



- Let $V_t \subset X_t$ be the subset of points such that $\lim_{\epsilon\to 0} \phi_{t,\epsilon}(v) = N$.
- V_t is a Lagrangian sphere called the vanishing cycle.

Vanishing cycles

Question

Which Lagrangian spheres arise as vanishing cycles of an algebraic nodal degeneration?

There are examples due to Corti and Smith of nullhomologous Lagrangian 3-spheres in $E \times \mathbb{CP}^1$ (where E is an Enriques surface) which do not arise as vanishing cycles. However, if we restrict to surfaces...

Theorem (Hind)

The space of Lagrangian spheres in a quadric surface is connected.

Recall that a quadric surface is symplectomorphic to the product of two spheres

$$Q \cong S^2 \times S^2$$

and there is an easy-to-spot *antidiagonal* Lagrangian sphere:

$$\bar{\Delta} = \{(x, -x) \in S^2 \times S^2 \; : x \in S^2\}$$

where $x \mapsto -x$ is the antipodal map on S^2.

- This is the vanishing cycle of a nodal degeneration.
- It is preserved by the switching symplectomorphism $(x, y) \mapsto (y, x)$.

Hind's theorem is proved using J-holomorphic curves. Let me give you an outline of the proof.

- Step 1: Prove that the group of symplectomorphisms of a quadric acts transitively on the space of Lagrangian spheres.
- Step 2: Use Gromov's theorem from last time that the group of symplectomorphisms of a quadric has two connected components, one containing the identity and one containing the switching map. We've already said that this latter component actually preserves $\bar{\Delta}$.

Hence the space of Lagrangian spheres is connected.

State of the art

Theorem

- *(E. 2010) The space of Lagrangian spheres in a given homology class in* $\mathbb{D}_2, \mathbb{D}_3, \mathbb{D}_4$ *is connected.*
- *(Borman-Li-Wu 2012) The group of symplectomorphisms acting trivially on homology acts transitively on Lagrangian spheres in a given homology class in any symplectic blow-up of* $\mathbb{CP}^2$*.*
- *(Corollary of Borman-Li-Wu and E. 2011) The Lagrangian spheres in* $\mathbb{D}_5$ *are precisely the vanishing cycles of algebraic nodal degenerations.*

Hind's theorem

To show that $\mathrm{Symp}(Q)$ acts transitively on Lagrangian spheres...

- Recall that a quadric surface Q has two rulings by holomorphic spheres.
- There are J-holomorphic rulings for any compatible almost complex structure J.
- Given a Lagrangian sphere L, **construct a J such that** any sphere from one of the J-holomorphic rulings intersects L transversely in a single point.
- (Symplectic tinkering:) Use this to define a symplectomorphism of the quadric taking the J-holomorphic rulings to the standard rulings and L to $\bar{\Delta}$.

The key step is the construction of J. This is achieved by neck-stretching!

The other theorems classifying vanishing cycles also use neck-stretching. The idea is to show that there are certain J_T-holomorphic curves which, for very large T, are disjoint from L, then use this to reduce the problem to Hind's theorem. The state of the art in that respect is the following theorem of Li and Wu:

Theorem (Li-Wu 2012)

Let L be a Lagrangian sphere in a symplectic 4-manifold (M, ω), and $A \in H_2(M; \mathbb{Z})$ with $A^2 \geq -1$. Suppose A is represented by a symplectic sphere C. Then C can be isotoped symplectically to another representative of A which intersects L minimally (i.e. transversely at $|[C] \cdot [L]|$ points).

There are related results for Lagrangian tori and higher genus surfaces due to Welschinger.

Summary

- We have seen that for Fano surfaces there is a close correlation between symplectic topology and algebraic geometry, both in terms of the symplectomorphism group and in terms of the space of Lagrangian spheres.
- We have seen more generally that symplectic topology has real consequences for real algebraic geometry and allows us to translate between different worlds (Riemannian/dynamical and algebro-geometric).
- We have encountered the idea of J-holomorphic curves (of central importance).
- We introduced neck-stretching, a technique which allows us to adapt out pseudoholomorphic curves to a Lagrangian submanifold.

This concludes the minicourse “Symplectic topology and algebraic geometry”. My next talk will be a research-level talk which will use neck-stretching and holomorphic discs to prove a smooth unknottedness theorem for Lagrangian tori in the simplest (and yet most enigmatic) of all symplectic manifolds, $\mathbb{C}^n$.

References

- M. Abreu, G. Granja and N. Kitchloo, "Compatible complex structures on symplectic rational ruled surfaces" arXiv:math/0610436 Duke Math. J. 148 (2009), no. 3, 539–600.
- S. Borman, T. J. Li and W. Wu, "Spherical Lagrangians via ball packings and symplectic cutting" (2012) arXiv:1211.5952.
- A. Corti and I. Smith, "Conifold transitions and Mori theory" arXiv:math/0501043 Math. Res. Lett. 12 (2005), no. 5-6, 767–778.
- Y. Eliashberg, A. Givental and H. Hofer, "Introduction to Symplectic Field Theory" arXiv:math/0010059 Visions in Mathematics, (GAFA 2000 Special Volume) Modern Birkhäuser Classics 2010, pp 560-673 (Theorem 1.7.5 is the Viterbo-Eliashberg theorem).
- J. D. Evans, "Lagrangian spheres in Del Pezzo surfaces" arXiv:0902.0540 J. Topol. (2010) 3(1):181–227.
- J. D. Evans, "Symplectic mapping class groups of some Stein and rational surfaces" arXiv:0909.5622 J. Sympl. Geom. (2011) 9(1):45–82.

References

- M. Gromov, "Pseudoholomorphic curves in symplectic manifolds" Invent. Math. 82 (1985), no. 2, 307–347.
- R. Hind, "Lagrangian spheres in $S^2 \times S^2$" arXiv:math/0311092 GAFA (2004) 14(2):303–318.
- J. Kollar, "The Nash conjecture for threefolds" arXiv:math/0009108 Electr. Res. Ann. of the Amer. Math. Soc. Volume 4, 63–73 and his papers Real Algebraic Threefolds I-IV.
- F. Lalonde and M. Pinsonnault "The topology of the space of symplectic balls in rational 4-manifolds" arXiv:math/0207096 Duke Math. J. Volume 122, Number 2 (2004), 347-397.
- T. J. Li and W. Wu, "Lagrangian spheres, symplectic surfaces and the symplectic mapping class group" arXiv:1012.4146 Geom. and Topol. 16 (2012) 1121-1169.

- F. Mangolte and J.-Y. Welschinger, "Do uniruled six-manifolds contain Sol Lagrangian submanifolds?" arXiv:1001.2927 Int. Math. Res. Not. IMRN 2012, no. 7, 1569–1602.
- P. Seidel, "Lectures on four-dimensional Dehn twists" arXiv:math/0309012 in *Symplectic 4-manifolds and algebraic surfaces*, 231–267, Lecture Notes in Math., Vol. 1938, Springer, Berlin, (2008).
- J.-Y. Welschinger, "Effective classes and Lagrangian tori in symplectic four-manifolds" arXiv:math/0701085 J. Sympl. Geom. (2007), 5(1):9–18.