Symplectic topology and algebraic geometry I: Symplectic mapping class groups

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Outline for today

Symplectic topology and algebraic geometry interact in many fruitful ways. I want to focus on the most concrete of these:

> a smooth complex projective variety is a symplectic manifold.

- If you're an **algebraic geometer**, I hope to give you a flavour of some simple techniques in symplectic topology
- If you're a **symplectic geometer** I hope to give you an idea of why it's helpful to understand some simple algebraic varieties.

The plan is to:

- Explain symplectic topology as a deformation invariant of a projective variety.
- Introduce the symplectic monodromy of a family of varieties (and thereby the symplectic mapping class group (SMCG)).
- Introduce Dehn twists: show that squared Dehn twists for algebraic surfaces are smoothly trivial.
- Prove a theorem of Seidel giving "lower bounds" on the size of the SMCG of a certain Del Pezzo surface.

This last point uses pseudoholomorphic curves to mimic certain constructions in projective geometry.

Fact

A smooth complex projective n-fold X is a symplectic $2n$-manifold:

- Inherits a Kähler 2-form ω from the ambient projective space $(\mathbb{CP}^N, \Omega)$.
- ω is closed ($\int_\sigma \omega = \int_{\sigma'} \omega$ if σ and σ' are homologous 2-cycles).
- ω is nondegenerate (a maximal isotropic subspace has dimension n).

Now we can talk about...

- **Lagrangian submanifolds:** Maximally isotropic submanifolds (n-dimensional submanifolds $\iota\colon L \to X$ such that $\iota^*\omega = 0$).
- **Symplectomorphisms:** Diffeomorphisms $\phi\colon X \to X$ such that $\phi^*\omega = \omega$.

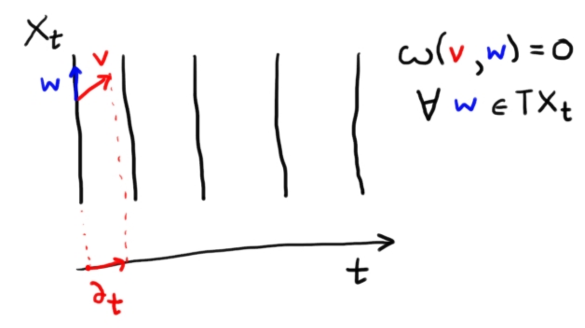
...for smooth projective varieties. Just like the diffeomorphism type, the symplectomorphism type of a smooth projective variety doesn't depend on the particular equations we use to cut it out:

Lemma (Deformation invariance)

Suppose $X_t \subset \mathbb{CP}^N$ is a (real) 1-parameter family of smooth projective varieties. Then they are all symplectomorphic.

Proof: Symplectic parallel transport.

- Start with a family $X \times \mathbb{R} \to \mathbb{CP}^N$.
- Let ω be the pullback of Ω to $X \times \mathbb{R}$ and ω_t be the pullback of ω to $X \times \{t\}$.
- Define v to be the unique vector field on $X \times \mathbb{R}$ such that:
  - its projection to $\mathbb{R}$ is ∂_t,
  - $\omega(v, w) = 0$ for all $w \in TX$.



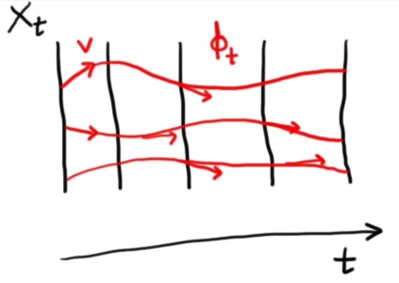
Proof: Symplectic parallel transport.

- Let $\phi_t \colon X \times \mathbb{R} \to X \times \mathbb{R}$ be the flow of v.

$$\left.\frac{d}{dt}\right|_{t=s} (\phi_t^* \omega) = \mathcal{L}_v \omega = d\iota_v \omega + \iota_v d\omega = 0$$

because

- $d\omega = 0$ implies $\iota_v d\omega = 0$,
- $\iota_v \omega = 0$ because $\omega(v, w) = 0$ for all $w \in TX$ and certainly $\omega(v, v) = 0$.



Monodromy I

Corollary (Monodromy representation)

Suppose that $\mathcal{X} \to M$ is a family of smooth projective subvarieties in $\mathbb{CP}^N$ containing X. There is a representation

$$\rho_{\mathsf{symp}} \colon \pi_1(M) \to \pi_0(\mathrm{Symp}(X)).$$

Here $\mathrm{Symp}(X)$ denotes the (infinite-dimensional Frèchet-Lie) group of all symplectomorphisms of X.

Definition

The group $\pi_0(\mathrm{Symp}(X))$ is called the symplectic mapping class group*.*

This generalises the monodromy representation

$$\rho \colon \pi_1(M) \to \mathrm{Aut}(H^*(X; \mathbb{Z})).$$

Dehn twists I

Suppose that $\pi\colon \mathcal{X} \to \Delta \subset \mathbb{C}$ is a family of projective varieties over the disc Δ where $X_0 = \pi^{-1}(0)$ is nodal and $\pi^{-1}(z)$ is smooth for $z \neq 0$. The monodromy around the unit circle is called a Dehn twist $\tau\colon X \to X$.

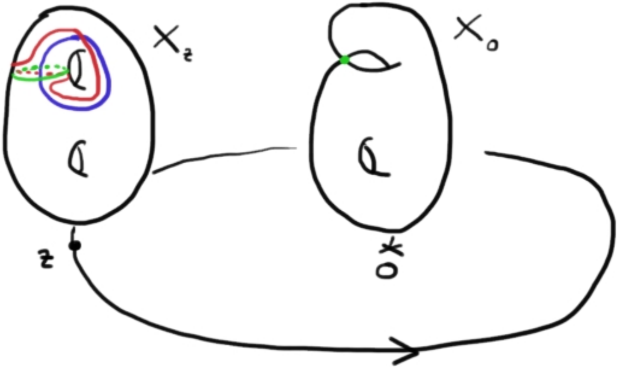
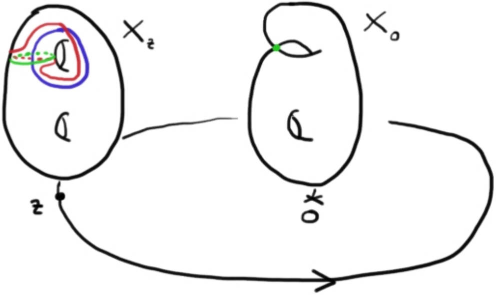


Figure : A Dehn twist.

Dehn twists II

Remark

- *The green circle in the picture is the set of points which gets crushed to the node if we parallel transport in towards the origin.*
- *This is called the* vanishing cycle*. In general it is a Lagrangian sphere.*
- *Actually you can assign a 'Dehn twist' to any Lagrangian sphere (it doesn't have to come from nodal degeneration).*



Dehn twists III

When $n = 2$, i.e. for algebraic surfaces, the Dehn twist acts as a reflection in $H^*(X; \mathbb{Z})$ (Picard-Lefschetz theorem). Hence

$$\tau^2 \in \ker \rho.$$

We will show:

- In fact, τ^2 is connected through diffeomorphisms to the identity diffeomorphism.
- Seidel showed that, in many cases, τ^2 represents a nontrivial element of $\pi_0(\mathrm{Symp}(X))$, i.e.
$$\rho_{\mathsf{symp}}(\tau^2) \neq 0.$$

We introduce the notation $\mathrm{Symp}_h(X)$ for the symplectomorphisms acting trivially on homology, so

$$\tau^2 \in \mathrm{Symp}_h(X).$$

Squared Dehn twists are smoothly trivial

Lemma (Kronheimer)

When X is an algebraic surface, the monodromy τ^2 is smoothly isotopic to the identity diffeomorphism.

Proof.

Let $\pi\colon \mathcal{X} \to \Delta$ be the nodal family. If $\mathrm{sq}\colon \Delta \to \Delta$ is the map $\mathrm{sq}(z) = z^2$ then define the pullback (base change)

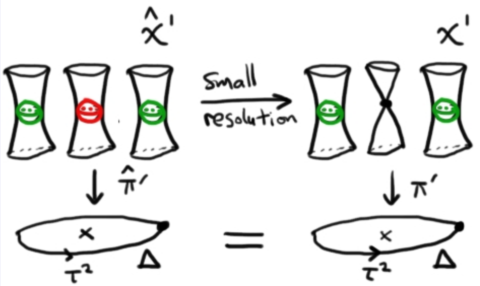
$$\begin{array}{ccc} \mathcal{X}' & \longrightarrow & \mathcal{X} \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ \Delta & \xrightarrow[\mathrm{sq}]{} & \Delta \end{array}$$

The space $\mathcal{X}'$ is a nodal 3-fold (with a single node in the fibre over 0) and the symplectic monodromy is τ^2. ...

Squared Dehn twists are smoothly trivial

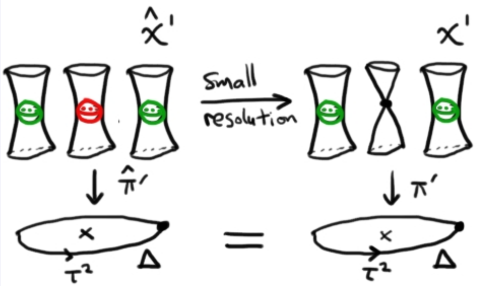
Proof (continued).

Take a small resolution $r\colon \check{\mathcal{X}} \to \mathcal{X}'$ (this replaces the node with a complex $\mathbb{CP}^1 \subset \check{\mathcal{X}}$ with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, red in the figure). We get a projection $\hat{\pi}'$ such that



commutes. …

Proof (continued).



Since the small resolution is an isomorphism away from the preimages of $0 \in \Delta$, the symplectic monodromy is still τ^2. However, the fibres of $\hat{\pi}'$ are all smooth (the nodal fibre has its node replaced by the red holomorphic sphere) so the monodromy is smoothly isotopic to the identity, i.e.

$$[\tau^2] = 1 \in \pi_0(\mathrm{Diff}(X)).$$

□

Squared Dehn twists can be symplectically nontrivial

- Crucially, the small resolution cannot be done symplectically - away from the central fibre, small pushoffs (in green!) of the new holomorphic sphere have vanishing symplectic area.
- In view of this, Seidel gave a nice argument to see that a squared Dehn twist can be nonzero in $\pi_0(\mathrm{Symp}(X))$ for $X = \mathbb{D}_5$:

Definition: $\mathbb{D}_5$

Let $X = \mathbb{D}_5$, the 5-point blow-up of $\mathbb{CP}^2$. This can be realised as a quadric-quadric intersection in $\mathbb{CP}^4$.

Theorem (Seidel)

If τ is the Dehn twist associated with a nodal degeneration of $\mathbb{D}_5$ then

$$\tau^2 \neq 1 \in \pi_0(\mathrm{Symp}(\mathbb{D}_5)).$$

Squared Dehn twists can be symplectically nontrivial

- There is a universal family $\mathcal{X} \to M$ of five-point blow-ups of $\mathbb{CP}^2$ over the configuration space M of ordered 5-tuples of general points in $\mathbb{CP}^2$ modulo $\mathbb{P}\,\mathrm{SL}(3,\mathbb{C})$.
- We need five points in general position (no three lie on a line) for the anticanonical map to be an embedding - otherwise the proper transform of a line through three points is contracted to a node by the anticanonical map.
- We get a symplectic monodromy map

$$F\colon \pi_1(M) \to \pi_0(\mathrm{Symp}_h(\mathbb{D}_5))$$

and we will show that F is injective. Since the loop defining τ^2 is nontrivial in $\pi_1(M)$ it will define a nontrivial symplectic mapping class.

Outline of proof

1. Construct a space $B\mathrm{Symp}_h(\mathbb{D}_5)$ with
$$\pi_1(B\mathrm{Symp}_h(\mathbb{D}_5)) = \pi_0(\mathrm{Symp}_h(\mathbb{D}_5)).$$
2. Construct a map $f\colon M \to B\mathrm{Symp}_h(\mathbb{D}_5)$ such that $F = \pi_1(f)$.
3. For a certain configuration space Y, construct a map
$$r\colon B\mathrm{Symp}_h(\mathbb{D}_5) \to Y.$$
4. Convince you that the composite $r \circ f$ gives an isomorphism
$$\pi_1(r \circ f)\colon \pi_1(M) \to \pi_1(Y)$$
so that $R = \pi_1(r)$ is a left-inverse for F.

Constructing $B\mathrm{Symp}_h(\mathbb{D}_5)$

- Given any topological group there is a space BG with $\pi_1(BG) = \pi_0(G)$.
- Just take a contractible free G-space EG: the quotient map $EG \to BG = EG/G$ is a fibration with fibre G.
- By the homotopy long exact sequence of the fibration and the fact that $\pi_i(E) = 0$ for all i, we get $\pi_i(BG) = \pi_{i-1}(G)$.

So we need a contractible free $\mathrm{Symp}_h(\mathbb{D}_5)$-space. The best example is the space of *compatible almost complex structures*.

Almost complex structures

Definition

Let $\mathcal{J}_\omega$ denote the space of ω-compatible almost complex structures on X, i.e. the space of endomorphisms

$$J\colon TX \to TX$$

such that $J^2 = -1$, $\omega(Jv, Jw) = \omega(v, w)$ and $\omega(v, Jv) > 0$ for all $v \neq 0$.

- Each J gives a metric $g(v, w) = \omega(v, Jw)$. If J is integrable and ω is the Kähler form then g is the Kähler metric.
- $\mathcal{J}_\omega$ is well-known to be contractible and certainly admits an action of $\mathrm{Symp}_h(X)$. We will see that this action is free when $X = \mathbb{D}_5$ (not true in general!).

J-holomorphic curves

The main ingredient is the notion of a J-holomorphic curve

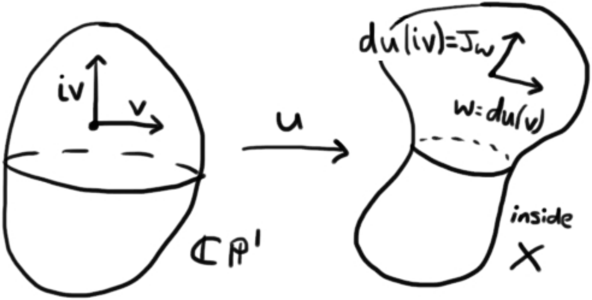


Figure : A J-holomorphic curve $u\colon \mathbb{CP}^1 \to X$.

Exceptional spheres

Theorem on -1-classes in a 4-manifold

Suppose that X is a symplectic 4-manifold and $E \in H_2(X;\mathbb{Z})$ is a homology class with $E^2 = -1$ and minimal symplectic area amongst all homology classes with positive area. If

- for **some** $J_0 \in \mathcal{J}_\omega$ the class E is represented by an embedded J_0-holomorphic sphere $E(J_0)$

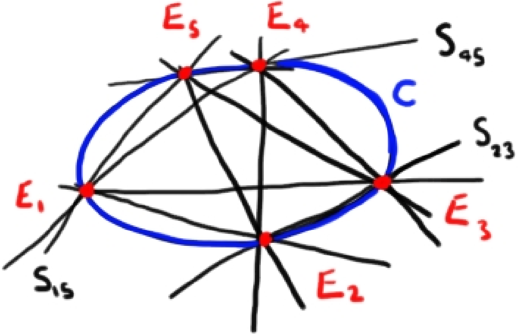
then

- for **all** $J \in \mathcal{J}_\omega$ the class is represented by a unique embedded J-holomorphic sphere $E(J)$.

- **Uniqueness:** Suppose two different J-curves represent E. When they intersect they intersect positively, but their intersection should compute $E^2 = -1$, which is negative.
- If E_1 and E_2 are two such classes with $E_1 \cdot E_2 = 1$ then by positivity of intersections, $E_1(J)$ and $E_2(J)$ intersect transversely at a single point.

Exceptional spheres

There are many exceptional spheres in $\mathbb{D}_5$:

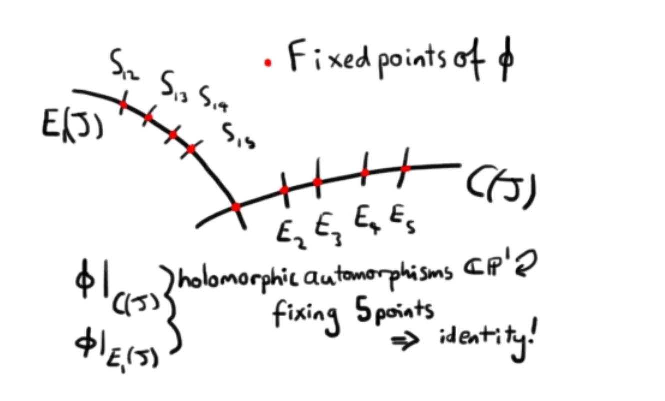


- the five blow-up curves $E_1, \ldots, E_5$,
- for each $1 \leq i < j \leq 5$, the proper transform S_{ij} of the line joining the points p_i and p_j,
- the proper transform C of the conic passing through all five.

The theorem on -1-classes tells us that they (and their intersection patterns) persist for arbitrary $J \in \mathcal{J}_\omega$.

Proof that $\mathrm{Symp}_h(\mathbb{D}_5)$ acts freely on $\mathcal{J}_\omega$.

If $\phi\colon \mathbb{D}_5 \to \mathbb{D}_5$ is a symplectomorphism acting trivially on homology and fixing a point $J \in \mathcal{J}_\omega$ then it must preserve (setwise) these J-holomorphic exceptional spheres. In particular it preserves their intersection points.



Proof that $\mathrm{Symp}_h(\mathbb{D}_5)$ acts freely on $\mathcal{J}_\omega$.

Since ϕ is a holomorphic automorphism of $C(J)$ and $E_1(J)$ fixing five points, it is the identity on each of these spheres. In particular it fixes the point

$$E_1(J) \cap C(J)$$

and acts as the identity on the tangent space at that point.

Since ϕ is an isometry of (ω, J), it commutes with the exponential map, so fixing a point and its tangent space implies $\phi = \mathrm{id}$. Therefore the $\mathrm{Symp}_h(\mathbb{D}_5)$-action on $\mathcal{J}_\omega$ is free. □

Outline of proof

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4. Convince you that the composite $r \circ f$ gives an isomorphism
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so that $R = \pi_1(r)$ is a left-inverse for F.

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A map $f\colon M \to B\mathrm{Symp}_h(\mathbb{D}_5)$

- A $\mathbb{P}\mathrm{SL}(3,\mathbb{C})$-orbit of five points $m \in M$ gives a complex structure on the blow-up X.
- The anticanonical embedding gives a symplectic structure on X, diffeomorphic to the standard one.
  - To get a symplectic structure we need a Fubini-Study form on $\mathbb{P}H^0(X, -K_X)$.
  - Such a form comes from a Euclidean metric on $H^0(X, -K_X)$.
  - There is a contractible space of choices of metric - let's pick one.
- Pulling back the complex structure along this diffeomorphism gives a point in $J(m) \in \mathcal{J}_\omega$.
- Two different identification diffeomorphisms differ by a symplectomorphism, hence $f(m) = [J(m)]$ is a well-defined map

$$f\colon M \to \mathcal{J}_\omega/\mathrm{Symp}_h(\mathbb{D}_5) = B\mathrm{Symp}_h(\mathbb{D}_5)$$

such that $\pi_1(f) = F$.

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Constructing r

Let $Y = \mathrm{Conf}_5^{\mathrm{ord}}(\mathbb{CP}^1)/\mathbb{P}\,\mathrm{SL}(2,\mathbb{C})$ be the configuration space of five ordered points on the sphere modulo holomorphic automorphisms.

Lemma/Definition

There is a map

$$r\colon B\mathrm{Symp}_h(\mathbb{D}_5) = \mathcal{J}_\omega/\mathrm{Symp}_h(\mathbb{D}_5) \to Y$$

which sends J to the configuration

$$(E_1(J) \cap C(J), \ldots, E_5(J) \cap C(J))$$

of five ordered intersection points on the conic (well-defined up to holomorphic automorphism).

Outline of proof

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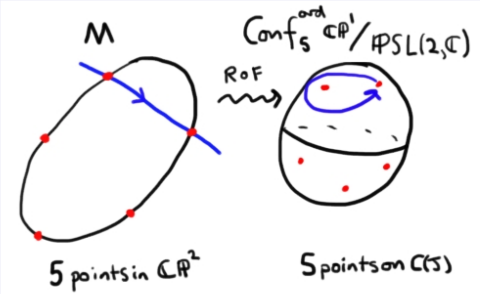
Completion of proof of Seidel's theorem.

Proving that $\pi_1(r \circ f)$ is an isomorphism only involves projective geometry. I won't do it, I'll just tell you that:

- $\pi_1(Y)$ is a quotient of the spherical 5-strand braid group by full twists,
- the monodromy τ^2 arises from a family in M where two points come together along the complex line which joins them,
- the corresponding path of configurations on the conic is an elementary braid on the sphere which is nontrivial.



Symplectic topology remembers algebraic geometry

- We were able to recover the fundamental group of the moduli space of Del Pezzo surfaces $\mathbb{D}_5$ as a subgroup of the symplectic mapping class group. Why?
- Because of the theory of J-holomorphic curves.
- When we took the symplectic manifold underlying a projective variety X, we thought we had sacrificed its rich theory of complex curves.
- Remarkably this survived in the form of J-holomorphic curve theory.

Probing $B\mathrm{Symp}$ using varieties

In general a family of projective varieties $\mathcal{X} \to M$ with fibre X gives a map $M \to B\mathrm{Symp}(X)$ and we can think of this as a finite-dimensional algebro-geometric probe into the mysterious space $B\mathrm{Symp}(X)$.

- How much of the topology of $B\mathrm{Symp}(X)$[1] can be captured in this way?
- Does $B\mathrm{Symp}(X)$ ever retract onto M?
- Is the symplectic mapping class group generated by Dehn twists?

[1] Note $B\mathrm{Symp}_h(X)$ is not always $\mathcal{J}_\omega/\mathrm{Symp}_h(X)$. One might ask: when does $\mathrm{Symp}_h(X)$ act freely on $\mathcal{J}_\omega$?

Gromov's theorem

Theorem (Gromov (1985))

If X is $\mathbb{CP}^2$ or the quadric surface $Q \subset \mathbb{CP}^3$ then the group $\mathrm{Symp}(X)$ retracts onto the subgroup of Kähler isometries.

- $\mathrm{Isom}(\mathbb{CP}^2) = \mathbb{P}\,\mathrm{U}(3)$ is connected.
- For $Q \cong S^2 \times S^2$, this group is $(SO(3) \times SO(3)) \rtimes \mathbb{Z}/2$ where the $\mathbb{Z}/2$ switches the two factors and the $SO(3)$-factors act by rotations of each factor S^2.
- This switching map is the monodromy τ of a nodal degeneration of Q, so actually for Q

$$[\tau^2] = 1 \in \pi_0(\mathrm{Symp}(Q)).$$

Gromov's theorem uses yet another amazing result from J-holomorphic curve theory:

Theorem

In a quadric surface Q, for an arbitrary $J \in \mathcal{J}_\omega$ there are two foliations of Q whose leaves are J-holomorphic spheres in the homology classes $A = [S^2 \times \{\star\}]$ and $B = [\{\star\} \times S^2]$. The A-spheres intersect the B-spheres in precisely one point transversely.

More general calculations

Theorem

In the following cases, $\pi_0(\mathrm{Symp}(X))$ is generated by Dehn twists:

- *(Lalonde-Pinnsonault 2004) When $X = \mathbb{D}_2$ (2-point blow-up of $\mathbb{CP}^2$) anticanonically embedded in $\mathbb{CP}^7$.*
- *(E. 2011) When $X = \mathbb{D}_k$, $k = 3, 4$, anticanonically embedded in $\mathbb{CP}^{9-k}$.*

In all these cases the squared Dehn twist is symplectically trivial and in fact $B\mathrm{Symp}_h(\mathbb{D}_k)$ retracts onto the moduli space of $k = 2, 3, 4$ points in $\mathbb{CP}^2$ modulo $\mathbb{P}\,\mathsf{SL}(3, \mathbb{C})$ (a single point!).

- *(E. 2011) For $\mathbb{D}_5$, the map $f : M \to B\mathrm{Symp}_h(\mathbb{D}_5)$ classifying the universal family of blow-ups is a homotopy equivalence.*

Question

This raises the question:

Question

Let (X,ω) be a symplectic manifold, $\mathcal{J}_\omega$ the space of compatible almost complex structures and $\mathcal{K}_\omega$ the space of integrable compatible almost complex structures. Does $\mathcal{J}_\omega$ retract $\mathrm{Symp}(X)$-equivariantly onto $\mathcal{K}_\omega$? Via a geometric flow?

Compare with the work of Abreu-Granja-Kitchloo (2009) in the case when X is a Hirzebruch surface (even or odd).

Conclusion

- The most important idea to take away is that there is a deformation-invariant theory of holomorphic curves, inherently symplectic but reflected in the enumerative geometry of the classical algebraic geometers.
- This is called Gromov-Witten theory.
- It turns even the most basic facts in algebraic geometry (rulings of a quadric surface, counts of conics through quintuplets of points in the plane) into powerful tools in symplectic geometry which give us a handle on complicated infinite-dimensional objects like $\mathrm{Symp}(X)$.