Symplectic topology and algebraic geometry I: Symplectic mapping class groups

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Outline for today

Symplectic topology and algebraic geometry interact in many fruitful ways. I want to focus on the most concrete of these:

A smooth complex projective variety is a symplectic manifold.

- If you’re an **algebraic geomter**, I hope to give you a flavour of some simple techniques in symplectic topology.
- If you’re a **symplectic geomter** I hope to give you an idea of why it’s helpful to understand some simple algebraic varieties.
The plan is to:

- Explain symplectic topology as a deformation invariant of a projective variety.
- Introduce the symplectic monodromy of a family of varieties (and thereby the symplectic mapping class group (SMCG)).
- Introduce Dehn twists: show that squared Dehn twists for algebraic surfaces are smoothly trivial.
- Prove a theorem of Seidel giving “lower bounds” on the size of the SMCG of a certain Del Pezzo surface.

This last point uses pseudoholomorphic curves to mimic certain constructions in projective geometry.
A smooth complex projective $n$-fold $X$ is a symplectic $2n$-manifold:

- Inherits a Kähler 2-form $\omega$ from the ambient projective space $(\mathbb{CP}^N, \Omega)$.
- $\omega$ is closed ($\int_\sigma \omega = \int_{\sigma'} \omega$ if $\sigma$ and $\sigma'$ are homologous 2-cycles).
- $\omega$ is nondegenerate (a maximal isotropic subspace has dimension $n$).
Now we can talk about...

- **Lagrangian submanifolds**: Maximally isotropic submanifolds ($n$-dimensional submanifolds $\iota: L \to X$ such that $\iota^*\omega = 0$).

- **Symplectomorphisms**: Diffeomorphisms $\phi: X \to X$ such that $\phi^*\omega = \omega$.

...for smooth projective varieties. Just like the diffeomorphism type, the symplectomorphism type of a smooth projective variety doesn’t depend on the particular equations we use to cut it out:

**Lemma (Deformation invariance)**

*Suppose $X_t \subset \mathbb{CP}^N$ is a (real) 1-parameter family of smooth projective varieties. Then they are all symplectomorphic.*
Proof: Symplectic parallel transport.

- Start with a family $X \times \mathbb{R} \to \mathbb{C}P^N$.
- Let $\omega$ be the pullback of $\Omega$ to $X \times \mathbb{R}$ and $\omega_t$ be the pullback of $\omega$ to $X \times \{t\}$.
- Define $v$ to be the unique vector field on $X \times \mathbb{R}$ such that:
  - its projection to $\mathbb{R}$ is $\partial_t$,
  - $\omega(v, w) = 0$ for all $w \in TX$. 

![Diagram of symplectic parallel transport](image.png)
Proof: Symplectic parallel transport.

Let $\phi_t : X \times \mathbb{R} \to X \times \mathbb{R}$ be the flow of $v$.

$$\left. \frac{d}{dt} \right|_{t=s} (\phi_t^* \omega) = \mathcal{L}_v \omega = d \iota_v \omega + \iota_v d\omega = 0$$

because

- $d\omega = 0$ implies $\iota_v d\omega = 0$,
- $\iota_v \omega = 0$ because $\omega(v, w) = 0$ for all $w \in TX$ and certainly $\omega(v, v) = 0$. 

![Diagram of symplectic parallel transport](image-url)
Corollary (Monodromy representation)

Suppose that $\mathcal{X} \to M$ is a family of smooth projective subvarieties in $\mathbb{CP}^N$ containing $X$. There is a representation

$$\rho_{\text{symp}} : \pi_1(M) \to \pi_0(\text{Symp}(X)).$$

Here $\text{Symp}(X)$ denotes the (infinite-dimensional Fréchet-Lie) group of all symplectomorphisms of $X$.

Definition

The group $\pi_0(\text{Symp}(X))$ is called the symplectic mapping class group.

This generalises the monodromy representation

$$\rho : \pi_1(M) \to \text{Aut}(H^*(X;\mathbb{Z})).$$
Dehn twists I

Suppose that $\pi: \mathcal{X} \to \Delta \subset \mathbb{C}$ is a family of projective varieties over the disc $\Delta$ where $X_0 = \pi^{-1}(0)$ is nodal and $\pi^{-1}(z)$ is smooth for $z \neq 0$. The monodromy around the unit circle is called a Dehn twist $\tau: X \to X$.

Figure: A Dehn twist.
Remark

- The green circle in the picture is the set of points which gets crushed to the node if we parallel transport in towards the origin.
- This is called the vanishing cycle. In general it is a Lagrangian sphere.
- Actually you can assign a ‘Dehn twist’ to any Lagrangian sphere (it doesn’t have to come from nodal degeneration).
Dehn twists III

When \( n = 2 \), i.e. for algebraic surfaces, the Dehn twist acts as a reflection in \( H^*(X; \mathbb{Z}) \) (Picard-Lefschetz theorem). Hence

\[
\tau^2 \in \ker \rho.
\]

We will show:

- In fact, \( \tau^2 \) is connected through diffeomorphisms to the identity diffeomorphism.
- Seidel showed that, in many cases, \( \tau^2 \) represents a nontrivial element of \( \pi_0(\text{Symp}(X)) \), i.e.

\[
\rho_{\text{symp}}(\tau^2) \neq 0.
\]

We introduce the notation \( \text{Symp}_h(X) \) for the symplectomorphisms acting trivially on homology, so

\[
\tau^2 \in \text{Symp}_h(X).
\]
Squared Dehn twists are smoothly trivial

Lemma (Kronheimer)

*When X is an algebraic surface, the monodromy $\tau^2$ is smoothly isotopic to the identity diffeomorphism.*

Proof.

Let $\pi: X \to \Delta$ be the nodal family. If $sq: \Delta \to \Delta$ is the map $sq(z) = z^2$ then define the pullback (base change)

$$
\begin{array}{ccc}
\mathcal{X}' & \longrightarrow & \mathcal{X} \\
\pi' \downarrow & & \downarrow \pi \\
\Delta & \longrightarrow & \Delta \\
& \text{sq} & \\
\end{array}
$$

The space $\mathcal{X}'$ is a nodal 3-fold (with a single node in the fibre over 0) and the symplectic monodromy is $\tau^2$. ...
Squared Dehn twists are smoothly trivial

Proof (continued).

Take a small resolution \( r : \check{\mathcal{X}} \to \mathcal{X}' \) (this replaces the node with a complex \( \mathbb{CP}^1 \subset \check{\mathcal{X}} \) with normal bundle \( \mathcal{O}(-1) \oplus \mathcal{O}(-1) \), red in the figure). We get a projection \( \hat{\pi}' \) such that

![Diagram showing the resolution process]

commutes.

...
Since the small resolution is an isomorphism away from the preimages of $0 \in \Delta$, the symplectic monodromy is still $\tau^2$. However, the fibres of $\hat{\pi}'$ are all smooth (the nodal fibre has its node replaced by the red holomorphic sphere) so the monodromy is smoothly isotopic to the identity, i.e.

$$[\tau^2] = 1 \in \pi_0(\text{Diff}(X)).$$
Squared Dehn twists can be symplectically nontrivial

- Crucially, the small resolution cannot be done symplectically - away from the central fibre, small pushoffs (in green!) of the new holomorphic sphere have vanishing symplectic area.
- In view of this, Seidel gave a nice argument to see that a squared Dehn twist can be nonzero in $\pi_0(\text{Symp}(X))$ for $X = D_5$:

**Definition: $D_5$**

Let $X = D_5$, the 5-point blow-up of $\mathbb{CP}^2$. This can be realised as a quadric-quadric intersection in $\mathbb{CP}^4$.

**Theorem (Seidel)**

*If $\tau$ is the Dehn twist associated with a nodal degeneration of $D_5$ then

$$\tau^2 \neq 1 \in \pi_0(\text{Symp}(D_5)).$$*
Squared Dehn twists can be symplectically nontrivial

- There is a universal family $\mathcal{X} \to M$ of five-point blow-ups of $\mathbb{CP}^2$ over the configuration space $M$ of ordered 5-tuples of general points in $\mathbb{CP}^2$ modulo $\mathbb{PSL}(3, \mathbb{C})$.
- We need five points in general position (no three lie on a line) for the anticanonical map to be an embedding - otherwise the proper transform of a line through three points is contracted to a node by the anticanonical map.
- We get a symplectic monodromy map

$$F : \pi_1(M) \to \pi_0(Symp_h(D_5))$$

and we will show that $F$ is injective. Since the loop defining $\tau^2$ is nontrivial in $\pi_1(M)$ it will define a nontrivial symplectic mapping class.
Outline of proof

1. Construct a space $BSymp_h(\mathbb{D}_5)$ with
   $$\pi_1(BSymp_h(\mathbb{D}_5)) = \pi_0(Symp_h(\mathbb{D}_5)).$$

2. Construct a map $f: M \to BSymp_h(\mathbb{D}_5)$ such that $F = \pi_1(f)$.

3. For a certain configuration space $Y$, construct a map
   $$r: BSymp_h(\mathbb{D}_5) \to Y.$$ 

4. Convince you that the composite $r \circ f$ gives an isomorphism
   $$\pi_1(r \circ f): \pi_1(M) \to \pi_1(Y)$$
   so that $R = \pi_1(r)$ is a left-inverse for $F$. 
Constructing $B\text{Symp}_h(\mathbb{D}_5)$

- Given any topological group there is a space $BG$ with $\pi_1(BG) = \pi_0(G)$.
- Just take a contractible free $G$-space $EG$: the quotient map $EG \to BG = EG/G$ is a fibration with fibre $G$.
- By the homotopy long exact sequence of the fibration and the fact that $\pi_i(E) = 0$ for all $i$, we get $\pi_i(BG) = \pi_{i-1}(G)$.

So we need a contractible free $\text{Symp}_h(\mathbb{D}_5)$-space. The best example is the space of compatible almost complex structures.
Almost complex structures

**Definition**

Let $J_\omega$ denote the space of $\omega$-compatible almost complex structures on $X$, i.e. the space of endomorphisms

$$J: TX \to TX$$

such that $J^2 = -1$, $\omega(Jv, Jw) = \omega(v, w)$ and $\omega(v, Jv) > 0$ for all $v \neq 0$.

- Each $J$ gives a metric $g(v, w) = \omega(v, Jw)$. If $J$ is integrable and $\omega$ is the Kähler form then $g$ is the Kähler metric.
- $J_\omega$ is well-known to be contractible and certainly admits an action of $\text{Symp}_h(X)$. We will see that this action is free when $X = \mathbb{D}_5$ (not true in general!).
**J-holomorphic curves**

The main ingredient is the notion of a $J$-holomorphic curve

**Figure**: A $J$-holomorphic curve $u: \mathbb{C}P^1 \to X$. 
Exceptional spheres

Theorem on $-1$-classes in a 4-manifold

Suppose that $X$ is a symplectic 4-manifold and $E \in H_2(X; \mathbb{Z})$ is a homology class with $E^2 = -1$ and minimal symplectic area amongst all homology classes with positive area. If

- for some $J_0 \in J_\omega$ the class $E$ is represented by an embedded $J_0$-holomorphic sphere $E(J_0)$

then

- for all $J \in J_\omega$ the class is represented by a unique embedded $J$-holomorphic sphere $E(J)$.

Uniqueness: Suppose two different $J$-curves represent $E$. When they intersect they intersect positively, but their intersection should compute $E^2 = -1$, which is negative.

If $E_1$ and $E_2$ are two such classes with $E_1 \cdot E_2 = 1$ then by positivity of intersections, $E_1(J)$ and $E_2(J)$ intersect transversely at a single point.
Exceptional spheres

There are many exceptional spheres in $\mathbb{D}_5$:

- the five blow-up curves $E_1, \ldots, E_5$,
- for each $1 \leq i < j \leq 5$, the proper transform $S_{ij}$ of the line joining the points $p_i$ and $p_j$,
- the proper transform $C$ of the conic passing through all five.
The theorem on $-1$-classes tells us that they (and their intersection patterns) persist for arbitrary $J \in J_\omega$.

**Proof that $\text{Symp}_h(D_5)$ acts freely on $J_\omega$.**

If $\phi : D_5 \to D_5$ is a symplectomorphism acting trivially on homology and fixing a point $J \in J_\omega$ then it must preserve (setwise) these $J$-holomorphic exceptional spheres. In particular it preserves their intersection points.
Proof that $\text{Symp}_h(D_5)$ acts freely on $\mathcal{J}_\omega$.

Since $\phi$ is a holomorphic automorphism of $C(J)$ and $E_1(J)$ fixing five points, it is the identity on each of these spheres. In particular it fixes the point

$$E_1(J) \cap C(J)$$

and acts as the identity on the tangent space at that point.

Since $\phi$ is an isometry of $(\omega, J)$, it commutes with the exponential map, so fixing a point and its tangent space implies $\phi = id$. Therefore the $\text{Symp}_h(D_5)$-action on $\mathcal{J}_\omega$ is free.
Outline of proof

1. Construct a space $B\text{Symp}_h(\mathbb{D}_5)$ with

$$\pi_1(B\text{Symp}_h(\mathbb{D}_5)) = \pi_0(\text{Symp}_h(\mathbb{D}_5)).$$

2. Construct a map $f: M \to B\text{Symp}_h(\mathbb{D}_5)$ such that $F = \pi_1(f)$.

3. For a certain configuration space $Y$, construct a map

$$r: B\text{Symp}_h(\mathbb{D}_5) \to Y.$$ 

4. Convince you that the composite $r \circ f$ gives an isomorphism

$$\pi_1(r \circ f): \pi_1(M) \to \pi_1(Y)$$

so that $R = \pi_1(r)$ is a left-inverse for $F$. 

Outline of proof

1. Construct a space $B\text{Symph}_h(\mathbb{D}_5)$ with $\pi_1(B\text{Symph}_h(\mathbb{D}_5)) = \pi_0(\text{Symph}_h(\mathbb{D}_5))$.

2. Construct a map $f: M \to B\text{Symph}_h(\mathbb{D}_5)$ such that $F = \pi_1(f)$.

3. For a certain configuration space $Y$, construct a map

$$r: B\text{Symph}_h(\mathbb{D}_5) \to Y.$$ 

4. Convince you that the composite $r \circ f$ gives an isomorphism

$$\pi_1(r \circ f): \pi_1(M) \to \pi_1(Y)$$

so that $R = \pi_1(r)$ is a left-inverse for $F$. 
A map $f : M \rightarrow B\text{Symp}_h(\mathbb{D}_5)$

- A $\mathbb{P}\text{SL}(3, \mathbb{C})$-orbit of five points $m \in M$ gives a complex structure on the blow-up $X$.
- The anticanonical embedding gives a symplectic structure on $X$, diffeomorphic to the standard one.
  - To get a symplectic structure we need a Fubini-Study form on $\mathbb{P}H^0(X, -K_X)$.
  - Such a form comes from a Euclidean metric on $H^0(X, -K_X)$.
  - There is a contractible space of choices of metric - let’s pick one.
- Pulling back the complex structure along this diffeomorphism gives a point in $J(m) \in J_\omega$.
- Two different identification diffeomorphisms differ by a symplectomorphism, hence $f(m) = [J(m)]$ is a well-defined map

$$f : M \rightarrow J_\omega/\text{Symp}_h(\mathbb{D}_5) = B\text{Symp}_h(\mathbb{D}_5)$$

such that $\pi_1(f) = F$. 
Outline of proof

1. Construct a space $B\text{Symp}_h(\mathbb{D}_5)$ with $\pi_1(B\text{Symp}_h(\mathbb{D}_5)) = \pi_0(\text{Symp}_h(\mathbb{D}_5))$.

2. Construct a map $f: M \to B\text{Symp}_h(\mathbb{D}_5)$ such that $F = \pi_1(f)$.

3. For a certain configuration space $Y$, construct a map

$$r: B\text{Symp}_h(\mathbb{D}_5) \to Y.$$ 

4. Convince you that the composite $r \circ f$ gives an isomorphism

$$\pi_1(r \circ f): \pi_1(M) \to \pi_1(Y)$$

so that $R = \pi_1(r)$ is a left-inverse for $F$. 
Constructing $r$

Let $Y = \text{Conf}^\text{ord}_5(\mathbb{CP}^1)/\mathbb{P}\text{SL}(2, \mathbb{C})$ be the configuration space of five ordered points on the sphere modulo holomorphic automorphisms.

**Lemma/Definition**

There is a map

$$r: B\text{Symp}_h(\mathbb{D}_5) = \mathcal{J}_\omega/\text{Symp}_h(\mathbb{D}_5) \to Y$$

which sends $J$ to the configuration

$$(E_1(J) \cap C(J), \ldots, E_5(J) \cap C(J))$$

of five ordered intersection points on the conic (well-defined up to holomorphic automorphism).
Outline of proof

1. Construct a space $BS\text{Symp}_h(D_5)$ with $\pi_1(B\text{Symp}_h(D_5)) = \pi_0(\text{Symp}_h(D_5))$.

2. Construct a map $f: M \to B\text{Symp}_h(D_5)$ such that $F = \pi_1(f)$.

3. For a certain configuration space $Y$, construct a map $r: B\text{Symp}_h(D_5) \to Y$.

4. Convince you that the composite $r \circ f$ gives an isomorphism $\pi_1(r \circ f): \pi_1(M) \to \pi_1(Y)$ so that $R = \pi_1(r)$ is a left-inverse for $F$. 
Completion of proof of Seidel’s theorem.

Proving that $\pi_1(r \circ f)$ is an isomorphism only involves projective geometry. I won’t do it, I’ll just tell you that:

- $\pi_1(Y)$ is a quotient of the spherical 5-strand braid group by full twists,
- the monodromy $\tau^2$ arises from a family in $M$ where two points come together along the complex line which joins them,
- the corresponding path of configurations on the conic is an elementary braid on the sphere which is nontrivial.
Symplectic topology remembers algebraic geometry

- We were able to recover the fundamental group of the moduli space of Del Pezzo surfaces $\mathbb{D}_5$ as a subgroup of the symplectic mapping class group. Why?
- Because of the theory of $J$-holomorphic curves.
- When we took the symplectic manifold underlying a projective variety $X$, we thought we had sacrificed its rich theory of complex curves.
- Remarkably this survived in the form of $J$-holomorphic curve theory.
In general a family of projective varieties $\mathcal{X} \to M$ with fibre $X$ gives a map $M \to B\text{Symp}(X)$ and we can think of this as a finite-dimensional algebro-geometric probe into the mysterious space $B\text{Symp}(X)$.

- How much of the topology of $B\text{Symp}(X)^1$ can be captured in this way?
- Does $B\text{Symp}(X)$ ever retract onto $M$?
- Is the symplectic mapping class group generated by Dehn twists?

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$^1$Note $B\text{Symp}_h(X)$ is not always $\mathcal{J}_\omega/\text{Symp}_h(X)$. One might ask: when does $\text{Symp}_h(X)$ act freely on $\mathcal{J}_\omega$?
Gromov’s theorem

**Theorem (Gromov (1985))**

*If $X$ is $\mathbb{CP}^2$ or the quadric surface $Q \subset \mathbb{CP}^3$ then the group $\text{Symp}(X)$ retracts onto the subgroup of Kähler isometries.*

- $\text{Isom}(\mathbb{CP}^2) = \mathbb{P} U(3)$ is connected.
- For $Q \cong S^2 \times S^2$, this group is $(SO(3) \times SO(3)) \rtimes \mathbb{Z}/2$ where the $\mathbb{Z}/2$ switches the two factors and the $SO(3)$-factors act by rotations of each factor $S^2$.
- This switching map is the monodromy $\tau$ of a nodal degeneration of $Q$, so actually for $Q$

$$[\tau^2] = 1 \in \pi_0(\text{Symp}(Q)).$$
Gromov’s theorem uses yet another amazing result from $J$-holomorphic curve theory:

**Theorem**

In a quadric surface $Q$, for an arbitrary $J \in \mathcal{J}_\omega$ there are two foliations of $Q$ whose leaves are $J$-holomorphic spheres in the homology classes $A = [S^2 \times \{\star\}]$ and $B = [\{\star\} \times S^2]$. The $A$-spheres intersect the $B$-spheres in precisely one point transversely.
More general calculations

**Theorem**

In the following cases, $\pi_0(\text{Symp}(X))$ is generated by Dehn twists:

- (Lalonde-Pinnsonault 2004) When $X = \mathbb{D}_2$ (2-point blow-up of $\mathbb{CP}^2$) anticanonically embedded in $\mathbb{CP}^7$.

- (E. 2011) When $X = \mathbb{D}_k$, $k = 3, 4$, anticanonically embedded in $\mathbb{CP}^{9-k}$.

In all these cases the squared Dehn twist is symplectically trivial and in fact $B\text{Symp}_h(\mathbb{D}_k)$ retracts onto the moduli space of $k = 2, 3, 4$ points in $\mathbb{CP}^2$ modulo $\mathbb{PSL}(3, \mathbb{C})$ (a single point!).

- (E. 2011) For $\mathbb{D}_5$, the map $f : M \to B\text{Symp}_h(\mathbb{D}_5)$ classifying the universal family of blow-ups is a homotopy equivalence.
This raises the question:

Let \((X, \omega)\) be a symplectic manifold, \(\mathcal{J}_\omega\) the space of compatible almost complex structures and \(\mathcal{K}_\omega\) the space of integrable compatible almost complex structures. Does \(\mathcal{J}_\omega\) retract \(\text{Symp}(X)\)-equivariantly onto \(\mathcal{K}_\omega\)? Via a geometric flow?

Compare with the work of Abreu-Granja-Kitchloo (2009) in the case when \(X\) is a Hirzebruch surface (even or odd).
Conclusion

- The most important idea to take away is that there is a deformation-invariant theory of holomorphic curves, inherently symplectic but reflected in the enumerative geometry of the classical algebraic geometers.
- This is called Gromov-Witten theory.
- It turns even the most basic facts in algebraic geometry (rulings of a quadric surface, counts of conics through quintuplets of points in the plane) into powerful tools in symplectic geometry which give us a handle on complicated infinite-dimensional objects like $\text{Symp}(X)$. 