Symplectic topology of some Stein and rational surfaces

Jonathan David Evans

Christ's College, Cambridge

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To Dad, for encouraging me to question everything.

Declaration: This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration except where specifically indicated in the text and bibliography.

I also state that my dissertation is not substantially the same as any that I have submitted for a degree or diploma or other qualification at any other University. I further state that no part of my dissertation has already been or is being concurrently submitted for any such degree, diploma or other qualification.

Summary of the thesis
“Symplectic topology of some Stein and rational surfaces”
by Jonathan David Evans:

A *symplectic manifold* is a $2n$-dimensional smooth manifold endowed with a closed, non-degenerate 2-form. This picks out the set of *Lagrangian submanifolds*, n-dimensional submanifolds on which the 2-form vanishes, and the group of *symplectomorphisms*, diffeomorphisms which preserve the symplectic form. In this thesis I study the homotopy type of the (compactly-supported) symplectomorphism group and the connectivity of the space of Lagrangian spheres for an array of symplectic 4-manifolds comprising some Stein surfaces and some Del Pezzo surfaces.

In part I of the thesis, concerning Stein surfaces, I calculate the homotopy type of the compactly-supported symplectomorphism group for $\mathbb{C}^* \times \mathbb{C}$ with its split symplectic form and $T^*\mathbb{RP}^2$ with its canonical symplectic form. More significantly, I show that the compactly-supported symplectomorphism group of the 4-dimensional A_n-Milnor fibre $\{x^2 + y^2 + z^{n+1} = 1\}$ is homotopy equivalent to a discrete group which injects naturally into the braid group on $n + 1$-strands.

In part II of the thesis, concerning Del Pezzo surfaces: I show that the isotopy class of a Lagrangian sphere in the monotone 2-, 3- or 4-point blow-up of $\mathbb{CP}^2$ is determined by its homology class; I calculate the homotopy type of the symplectomorphism group for the monotone 3-, 4- and 5-point blow-ups of $\mathbb{CP}^2$.

The calculations of homotopy groups of symplectomorphism groups rely on nothing more than the standard technology of pseudoholomorphic curves and some involved topological arguments to prove the fibration property of various maps between infinite-dimensional spaces. The new idea is the compactification of the Milnor fibres by a configuration of holomorphic spheres which puts the calculation in a context familiar from the work of Lalonde-Pinsonnault and Abreu.

The classification of Lagrangian spheres is based on an argument of Richard Hind in the case of $S^2 \times S^2$ using the technique of neck-stretching from symplectic field theory. The analysis is more involved and crucially uses a transversality theorem for the asymptotic evaluation map from a moduli space of simple punctured holomorphic curves to a Morse-Bott space of Reeb orbits.

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Contents

Chapter 1

Introduction

In this thesis I study some four-dimensional symplectic manifolds, focusing on their symplectomorphisms and Lagrangian submanifolds. I will give a complete description of the homotopy type of the symplectomorphism group in all cases and prove connectedness for the space of homologous Lagrangian spheres in some cases.

The introduction comprises:

1. An explanation of those properties of symplectic manifolds which concern us,
2. A description of the specific symplectic manifolds under study and a precise statement of the results obtained,
3. A commentary on these results and some more general motivating comments,
4. An outline of the rest of the thesis,
5. A breakdown of the content, indicating which parts are merely recapitulations of standard material and which are original.

All symplectic manifolds we consider will be four-dimensional.

1.1 Properties of symplectic manifolds

We recall that a symplectic manifold is a pair (M, ω) where M is a smooth manifold and ω is a closed, non-degenerate 2-form on M.

1.1.1 Symplectomorphism groups

A symplectomorphism of (M, ω) is a diffeomorphism of M which preserves ω. For fixed (M, ω) the set of these forms a group under composition which can be made into a Fréchet-Lie group. When M is 2-dimensional it is a classical result that the symplectomorphism group is a retract of the diffeomorphism group, and the topology of diffeomorphism groups of 2-manifolds is well-understood [13]. It was Gromov who first calculated the homotopy type of a symplectomorphism group for a 4-dimensional symplectic manifold.

Theorem 1 ([16]). *The symplectomorphism group of* $(S^2 \times S^2, \omega \oplus \omega)$ *deformation retracts onto the subgroup* $(SO(3) \times SO(3)) \rtimes \mathbb{Z}/2$ *of Kähler isometries.*

This remarkable theorem was proved using the theory of pseudoholomorphic curves, developed by Gromov in the same paper. The special property of $S^2 \times S^2$ which makes the symplectomorphism group amenable to calculations is the following:

Theorem 2 ([16]). *If J is an almost complex structure on $X = S^2 \times S^2$ which is compatible with the product symplectic form ω then there exist J-holomorphic foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ of X whose leaves are spheres in the homology classes $S^2 \times \{pt\}$ and $\{pt\} \times S^2$ respectively.*

Since J-holomorphic curves intersect positively, a leaf of $\mathcal{F}_1$ and a leaf of $\mathcal{F}_2$ intersect transversely in a single point. This allows one to construct diffeomorphisms conjugating the foliation pairs coming from different almost complex structures and a clever Moser-type argument is used to deduce theorem 1. Later authors used pseudoholomorphic curve techniques to compute the homotopy type of symplectomorphism groups of further examples:

- $\mathrm{Symp}(\mathbb{CP}^2, \omega)$ is homotopy equivalent to $\mathbb{P}U(3)$, where ω is the Fubini-Study form (due to Gromov, [16]),
- $\mathrm{Symp}(\mathbb{D}_1, \omega)$ is homotopy equivalent to $U(2)$, where $\mathbb{D}_1$ is the 1-point blow-up of $\mathbb{CP}^2$ and ω is the anticanonical Kähler form, (due to Gromov, [16]),
- $\mathrm{Symp}(\mathbb{D}_2, \omega)$ is weakly homotopy equivalent to $T^2 \rtimes \mathbb{Z}/2$, where $\mathbb{D}_2$ is the 2-point blow-up of $\mathbb{CP}^2$ and ω is the anticanonical Kähler form (due to Lalonde-Pinsonnault, [25]),
- $\mathrm{Symp}_c(\mathbb{C}^2)$ is contractible, where ω is the product symplectic form (due to Gromov, [16]),
- $\mathrm{Symp}_c(T^*S^2)$ is weakly homotopy equivalent to $\mathbb{Z}$, where ω is the canonical symplectic form (due to Seidel, see [33]).

Here, Symp_c denotes the group of compactly-supported symplectomorphisms, topologised as a direct limit over compact supports. The last two cases work because $\mathbb{C}^2$ and T^*S^2 embed into $S^2 \times S^2$ and inherit foliations by pseudoholomorphic discs from the foliations of theorem 2. In cases such as $\mathbb{D}_2$, one does not have global foliations by pseudoholomorphic curves for topological reasons. However, one has well-behaved -1-spheres (exceptional spheres from the blow-up) which persist under deformations of the almost complex structure. One can find a configuration of these -1-spheres whose complement is symplectomorphic to $\mathbb{C}^2$. Contractibility of $\mathrm{Symp}_c(\mathbb{C}^2)$ now reduces the study of $\mathrm{Symp}(\mathbb{D}_2)$ to that of 'symplectomorphisms' of the configuration of spheres.

This style of argument reflects our understanding of symplectic manifolds. Donaldson [12] proved that any symplectic manifold contains a codimension 2 symplectic submanifold (representing some multiple of the Poincaré dual homology class to the cohomology class of the symplectic form) whose complement is symplectomorphic to a Stein manifold. This is a symplectic analogue of Bertini's theorem in complex projective geometry which says one can take a transverse hyperplane section of a projective variety. For this reason, codimension 2 symplectic submanifolds representing a multiple of $\mathrm{PD}[\omega]$ are often called *symplectic hypersurfaces*. This point of view has yielded many results in symplectic topology, for example [4], [11] and motivates the constructions in this thesis. The main difference is that we will use a *singular* hypersurface (a configuration of spheres) because of its good properties under deformation of the almost complex structure.

Remark 1.1.2. *One glaring feature of the list above is the absence of any manifolds of dimension bigger than four. There are special properties of pseudoholomorphic curves in four dimensions (see appendix B) which allow one to perform calculations. This thesis is based on pseudoholomorphic curve theory and so our examples are all four-dimensional.*

1.1.3 Lagrangian spheres

Recall that an embedded submanifold $\iota : L \to M$ of a $2n$-dimensional symplectic manifold (M, ω) is called *Lagrangian* if $\dim L = n$ and $\iota^*\omega = 0$. Isotopy of Lagrangian submanifolds through Lagrangian submanifolds is called Lagrangian isotopy. These form an extremely important class of submanifolds. For four-dimensional symplectic manifolds $n = 2$ so Lagrangian submanifolds must be 2-manifolds. We specialise to those which are diffeomorphic to the 2-sphere. Aside from basic complications which non-simply connectedness bring, it seems that tori and higher genus Lagrangian surfaces are much more subtle objects than Lagrangian spheres.

To a Lagrangian sphere L in a symplectic 4-manifold one can associate (canonically up to isotopy) a symplectomorphism supported in a neighbourhood of the sphere, called the *Dehn twist* in L. See [37] for a definition. Lagrangian isotopic spheres L_1 and L_2 yield isotopic Dehn twists, so the topology of the space of Lagrangian spheres is evidently related to the topology of the group of symplectomorphisms.

In [19], Richard Hind made a breakthrough in our understanding of Lagrangian spheres using techniques from symplectic field theory.

Theorem 3 ([19]). *Any two Lagrangian spheres in* $(S^2 \times S^2, \omega \oplus \omega)$ *are Lagrangian isotopic, i.e. the space of Lagrangian spheres is connected.*

The simplest example of a Lagrangian in this manifold is the antidiagonal sphere

$$L = \left\{(x, -x) \in S^2 \times S^2\right\}$$

and the Dehn twist in L is isotopic to the symplectomorphism

$$(x, y) \mapsto (y, x).$$

Recall that by Gromov's theorem (theorem 1), $\mathrm{Symp}(S^2 \times S^2)$ has two components and that this Dehn twist is in the non-identity component. This is a very clear illustration of the relation alluded to above between the topology of the symplectomorphism group and the topology of the space of Lagrangian spheres via Dehn twists.

Theorems like Hind's or theorem E proved in this thesis are probably the exception rather than the rule, in that they are *classification results*: knowing the homology class of a Lagrangian sphere determines it up to isotopy. This is absolutely untrue in general, as demonstrated by Seidel in [34]. See the comments in section 1.3.

Hind uses the existence of foliations by holomorphic curves on $S^2 \times S^2$. In our examples we will not have that luxury, so again we resort to the picture of a symplectic manifold split into a symplectic hypersurface and its complement. It is clear from the definition that Lagrangian submanifolds satisfy $\omega([L]) = 0$, i.e. that $PD(\omega) \cdot [L] = 0$ under the intersection product on homology. The idea will be to isotope L until its geometric intersection with the symplectic hypersurface is empty.

In fact, it is a theorem of Auroux, Muñoz and Presas [2] that for some $k >> 0$ there is a symplectic hypersurface H Poincaré dual to $k[\omega]$ for which $L \cap H = \emptyset$. They use the same 'approximately holomorphic' techniques as Donaldson used in [12]. We want a sharper theorem which allows us to disjoin our Lagrangian from some particular hypersurface with small k. In fact we use a configuration of intersecting pseudoholomorphic spheres which persists under deformations of the almost complex structure and apply neck stretching techniques from symplectic field theory to construct a disjoining isotopy.

We will choose our configuration of intersecting pseudoholomorphic spheres so that its complement will be symplectomorphic to a neighbourhood of the zero-section in T^*S^2. This choice will depend only on the choice of homology class of our Lagrangian sphere, so we show that any two homologous Lagrangian spheres can be isotoped into this subset of the ambient manifold. Since Hind has shown that there is a unique Lagrangian sphere up to isotopy in a neighbourhood of the zero-section in T^*S^2 (see [18]), we will be able to deduce that homologous Lagrangians are isotopic.

1.2 Examples under study

Our examples fall into two categories: Stein surfaces and Del Pezzo surfaces.

1.2.1 Stein surfaces

Stein manifolds are defined in section 2.1 below. They are well-behaved but non-compact symplectic manifolds. The canonical examples are complex affine varieties. It should be clear from the explanations above that in order to understand closed symplectic manifolds we have to understand their codimension 0 Stein submanifolds arising as complements of Donaldson's symplectic hypersurfaces. However, Stein manifolds are also interesting in their own right. A very beautiful family of Stein manifolds are the A_n-Milnor fibres

$$\left\{(x,y,z)\in\mathbb{C}^3 : x^2+y^2+z^{n+1}=1\right\}$$

with the symplectic form inherited from the standard symplectic form on $\mathbb{C}^3$. These spaces can be thought of as *plumbings* of n copies of T^*S^2 arranged in an A_n-chain.

Theorem A. *Let W be the A_n-Milnor fibre. Then the group of compactly supported symplectomorphisms of (W,ω) is weakly homotopy equivalent to its group of components. This group of components injects homomorphically into the braid group Br_{n+1} of $n+1$-strands on the disc.*

We will comment on the slightly mysterious statement of this theorem in section 1.3 below. The other examples are $T^*\mathbb{RP}^2$ and $\mathbb{C}^*\times\mathbb{C}$.

Theorem B. *Let ω be the canonical symplectic form on the cotangent bundle of $\mathbb{RP}^2$. The group of compactly supported symplectomorphisms $Symp_c(T^*\mathbb{RP}^2)$ is weakly homotopy equivalent to $\mathbb{Z}$.*

Theorem C. *Let ω be the product symplectic form on $\mathbb{C}^*\times\mathbb{C}$. The group of compactly supported symplectomorphisms $Symp_c(\mathbb{C}^*\times\mathbb{C})$ is weakly contractible.*

These are perhaps less interesting but are useful for understanding both the Milnor fibres and later the Del Pezzo surfaces. The calculation for $\mathbb{C}^*\times\mathbb{C}$ is provides a model for understanding the more terminologically-cluttered calculation for the Milnor fibres.

For these examples we will not say anything about the Lagrangian spheres, though Hind claims to have a classification of Lagrangian spheres in the A_2-Milnor fibre (see [18]).

1.2.2 Del Pezzo surfaces

These are monotone blow-ups of the complex projective plane in eight or fewer points. We will denote the n-point blow-up by $\mathbb{D}_n$ and consider the symplectomorphism group of $\mathbb{D}_3$, $\mathbb{D}_4$ and $\mathbb{D}_5$. The reason for this is the following:

- In $\mathbb{D}_3$ we can find a chain of five -1-spheres whose complement is biholomorphic to $\mathbb{C}^* \times \mathbb{C}$.
- In $\mathbb{D}_4$ we can find a configuration of six -1-spheres whose complement is biholomorphic to $\mathbb{C}^2$.
- In $\mathbb{D}_5$ we can find a configuration of six -1-spheres whose complement is biholomorphic to $T^*\mathbb{RP}^2$ (recall that $T^*\mathbb{RP}^2$ is realised as the complement of a conic curve in $\mathbb{CP}^2$ and that there is a unique conic through five points - our configuration of spheres consists of the proper transform of the conic and the five exceptional spheres. This picture was inspired by [37]).

Once we blow-up more than five times, I cannot find a configuration of spheres whose complement is something I understand.

Symplectomorphism groups

If Symp_0 denotes the group of symplectomorphisms acting trivially on homology, we will prove that

Theorem D. *If $\mathbb{D}_n$ denotes the monotone n-point blow-up of $\mathbb{CP}^2$ in n symplectic balls then*

1. *$Symp_0(\mathbb{D}_3)$ is weakly homotopy equivalent to T^2,*
2. *$Symp_0(\mathbb{D}_4)$ is weakly contractible,*
3. *$Symp_0(\mathbb{D}_5)$ is weakly homotopy equivalent to $Diff^+(S^2, 5)$, the group of orientation-preserving diffeomorphisms of S^2 preserving five points.*

The author learned recently that Martin Pinsonnault had independently obtained similar (unpublished) results [30].

Lagrangian spheres

For $\mathbb{D}_2$, $\mathbb{D}_3$ and $\mathbb{D}_4$ we also consider the Lagrangian isotopy problem:

Theorem E. *If L_1 and L_2 are homologous Lagrangian spheres in any one of $\mathbb{D}_2$, $\mathbb{D}_3$ or $\mathbb{D}_4$ then they are Lagrangian isotopic.*

The automorphisms of $H_2(\mathbb{D}_n, \mathbb{Z})$ preserving the intersection form are precisely those arising from the Picard-Lefschetz formula for Dehn twists in the homology classes of Lagrangian spheres. The connectedness of Symp_0 and the connectedness of the space of homologous Lagrangians therefore gives us a precise correspondence between Lagrangian spheres and symplectomorphisms via Dehn twists in the cases $\mathbb{D}_2$, $\mathbb{D}_3$ and $\mathbb{D}_4$ (recall that $\mathrm{Symp}_0(\mathbb{D}_2) \simeq T^2$ by [25]). I could not find a convenient configuration of divisors from which to disjoin Lagrangian spheres in $\mathbb{D}_5$, which is where one notices something interesting in the symplectomorphism group. In fact, it is certainly true that there are homologous Lagrangians in $\mathbb{D}_5$ which are not Lagrangian isotopic. For more on this, see the comments in section 1.3 below.

1.3 Comments on results

1.3.1 Milnor fibres

Theorem A is not as strong as we would like. The injective homomorphism we construct into the braid group is probably also a surjection. As we remarked above, there is an A_n-chain of Lagrangian spheres in the A_n-Milnor fibre and the Dehn twists around these are likely to correspond to standard generators of the braid group, giving preimages for all the generators. I have been unable to prove this for technical reasons. However, we can still deduce something more about the symplectic mapping class group $\pi_0(\mathrm{Symp}_c(W))$ in the case $n \geq 4$. Khovanov and Seidel demonstrated [24] using Fukaya categories that the braid group Br_n always injects into $\pi_0(\mathrm{Symp}_c(W))$. The composition of their injection with ours is therefore a homomorphic injection $K : \mathrm{Br}_n \to \mathrm{Br}_n$. While the braid group is not co-Hopfian, it is known [3] that, when $n \geq 4$, all such injections are of the form

$$\sigma_i \mapsto z^\ell \left(h^{-1}\sigma_i h\right)^{\pm 1}$$

where σ_i are the usual generators, h is a homeomorphism of the marked disc, $z \in Z(\mathrm{Br}_n)$ is a full-twist and $\ell \in \mathbb{Z}$. In particular, we know that the subgroup $K(\mathrm{Br}_n)$ is of finite index and that all intermediate subgroups between $K(\mathrm{Br}_n)$ and Br_n are isomorphic to Br_n, in particular the group $\pi_0(\mathrm{Symp}_c(W))$ is abstractly isomorphic to Br_n.

1.3.2 $\mathbb{D}_5$

Seidel [37] has also obtained results in the direction of theorem D, specifically exhibiting a subgroup of $\pi_0(\mathrm{Symp}_0(\mathbb{D}_5))$ isomorphic to $\pi_0(\mathrm{Diff}^+(S^2, 5))$. These results were the starting point for this paper and will be reviewed in section 6.1.

Notice that this subgroup is generated by Dehn twists in Lagrangian spheres. Since isotopic Lagrangian spheres must necessarily have isotopic Dehn twists, the existence of homologous Lagrangians with non-isotopic Dehn twists implies the existence of non-isotopic homologous Lagrangians. We call this phenomenon *Lagrangian knotting*. We could rephrase theorem E by saying there is no Lagrangian knotting in $\mathbb{D}_2$, $\mathbb{D}_3$ or $\mathbb{D}_4$.

As an indication of how general the phenomenon of Lagrangian knotting is, and therefore how unusually simple our examples $\mathbb{D}_2$, $\mathbb{D}_3$ and $\mathbb{D}_4$ are, Seidel proved the following beautiful result:

Theorem 4 (Seidel, [35])**.** *Let W be the A_n-Milnor fibre ($n > 2$). In each homology class containing a Lagrangian 2-sphere there are infinitely many smoothly isotopic Lagrangian 2-spheres which are pairwise knotted.*

Suppose (M, ω) contains an A_n-chain of Lagrangian spheres. By Weinstein's Lagrangian neighbourhood theorem, these have a neighbourhood symplectomorphic to a neighbourhood of the basic A_n-chain in the A_n-Milnor fibre, so locally Lagrangian knotting occurs. Indeed, in our examples it is possible to find A_3 and A_4 configurations of Lagrangian spheres inside $\mathbb{D}_3$ and $\mathbb{D}_4$ respectively and a corollary of theorem E is that although two homologous Lagrangian spheres in these configurations are isotopic, the isotopy must pass through Lagrangian spheres which leave the A_n-neighbourhoods.

1.3.3 Where the answers come from

Symplectomorphism groups

The symplectomorphism groups are handled by finding a configuration of pseudoholomorphic spheres whose complement is a well-understood Stein manifold. Various long exact sequences now reduce the computation to understanding the group of symplectomorphisms of components of this configuration (fixing the intersection points) and the group of gauge transformations of the normal bundles to these components (equal to the identity at the intersection points). See section 2.3 for clarification.

In the non-compact cases like the Milnor fibres where there are no such configurations of pseudoholomorphic spheres one first compactifies the space with a divisor C'. The group of compactly-supported symplectomorphisms of the Milnor fibre is then realised as the group of symplectomorphisms of the compactification which fix a neighbourhood of

C'. The compactification procedure raises issues with the symplectic form being used: in particular, the complement of C' has finite volume, while the Milnor fibre has infinite volume. These issues are ironed out in section 2.1.

The fact that all answers come from symplectomorphism groups of configurations of spheres explains the appearance of braid groups.

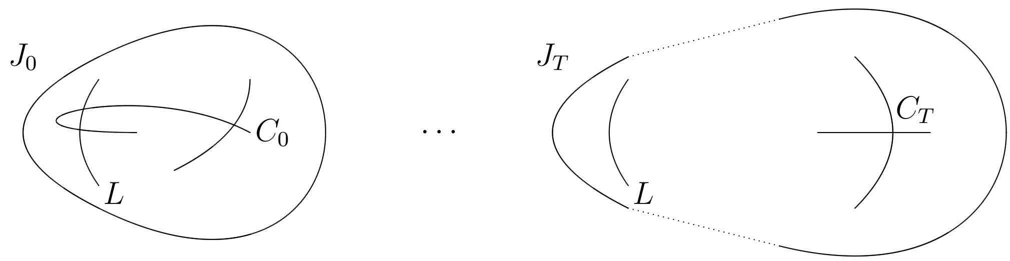
Lagrangian spheres

Theorem E is proved by first showing that one can find a Lagrangian isotopy taking any 2-sphere L to another sphere L' which is disjoint from some set of divisors. The divisors are chosen so that their complement is symplectomorphic to (a compact subset of) T^*S^2, whereupon Hind's theorem guarantees that any two such spheres are Lagrangian isotopic. Achieving disjointness from divisors requires the technology of symplectic field theory and takes up most of the paper.

The idea behind proving disjointness is to find a family of almost complex structures $\{J_t\}_{t=0}^T$ and a family C_t of J_t-holomorphic curves representing the relevant configuration of divisors such that:

- J_0 is the standard complex structure and C_0 is the standard configuration of divisors,
- C_T is disjoint from L.

Then it is not hard (see section 5.6) to construct a disjoining isotopy of L from C_0. The almost complex structures J_t are obtained by a process called "stretching the neck" around L (see diagram 1.1 below). The J_t-holomorphic curves are proven to exist in section 4.3. The behaviour of C_t as $t \to \infty$ is analysed using symplectic field theory and it is shown in sections 5.4 and 5.5 that for large t, C_t must be disjoint from L.



(1.1)

1.3.4 Why we expect these answers

Complex geometry

Let $(\chi, \mathfrak{L}) \to S$ be a family of polarised smooth complex projective varieties with a choice of fibrewise hermitian metric on the polarisation and write $(\chi_s, \mathfrak{L}_s)$ for the fibre over $s \in S$. Let S_N be the locus of $s \in S$ for which $\mathfrak{L}_s^{\otimes N}$ is very ample. We obtain a fibrewise symplectic form $\{\omega_s^N\}_{s\in S_N}$ by pulling back the Fubini-Study form along the natural embedding

$$\begin{aligned}\chi_s &\to \mathbb{P}(H^0(\chi_s, \mathfrak{L}|_{\chi_s}^{\otimes N})^\vee \\ x &\mapsto \mathbb{P}(\mathrm{ev}_x : \sigma \mapsto \sigma(x))\end{aligned}$$

and rescaling by $1/N$. The cohomology class of this form is constant and the symplectic form varies smoothly so using Moser's argument we can perform symplectic parallel transport along paths in S. Fixing a basepoint $0 \in S_N$ and writing $(X,\omega) = (\chi_0, \omega_0^N)$ this gives a map

$$\iota_N : \Omega_0 S_N \to \mathrm{Symp}(X,\omega)$$

(where Ω_0 denotes the based loop space) which induces maps

$$\pi_i(S_N) \to \pi_{i-1}(\mathrm{Symp}(X,\omega)) \cong \pi_i(B\mathrm{Symp}(X,\omega))$$

This gives us a good candidate for non-trivial homotopy classes of the symplectomorphism group. This argument can also be made to work for Stein manifolds by taking care with parallel transport maps.

Remark 1.3.5. *Varying N it will sometimes happen that S_N changes. If $0 \in S_N$ then $0 \in S_{NM}$ for all $M > 0$ and it is easy to show that ω_0^{NM} and ω_0^N are symplectomorphic. However, it is possible that ι_N and $\iota_{NM}|_{\Omega_0 S_N}$ are not conjugated by this symplectomorphism. It would be interesting to find examples where this failure of ampleness gave rise to a non-trivial symplectomorphism.*

In the cases studied in this thesis it is easy to write down a family χ of algebraic varieties for which

$$\pi_i(S) \to \pi_i(B\mathrm{Symp}(X,\omega))$$

is an injection for all i. It was natural to ask which of these maps are isomorphisms.

For example if X is the monotone symplectic Del Pezzo surface $\mathbb{D}_5$ then let $S = \mathrm{Conf}_5^{\mathrm{ord,gen}}(\mathbb{CP}^2)$ denote the space of ordered configurations of five points in general position in $\mathbb{CP}^2$ (modulo projective linear transformations) and let $\chi \to S$ be the associated universal family of blow-ups of $\mathbb{CP}^2$ in five points. To a configuration $c \in S$ we associate a configuration c' of five points in $\mathbb{CP}^1$ (modulo the Möbius group) by observing that any five points in general position lie on a (unique, smooth) conic and defining c' to be the five points considered as a configuration on the conic $\mathbb{CP}^1$. The corresponding map $\mathrm{Conf}_5^{\mathrm{ord,gen}}(\mathbb{CP}^2) \to \mathrm{Conf}_5^{\mathrm{ord}}(\mathbb{CP}^1)$ is a homotopy equivalence. It was known to Seidel (see [37]) that the induced maps

$$\pi_i(\mathrm{Conf}_5^{\mathrm{ord}}(\mathbb{CP}^1)) \cong \pi_i(S) \to \pi_i(B\mathrm{Symp}(\mathbb{D}_5))$$

are injections. Theorem D.3 implies that they are isomorphisms. Therefore in particular, any symplectic mapping class $m \in \pi_0(\mathrm{Symp}(\mathbb{D}_5))$ is represented by a symplectic parallel transport map associated to a loop in the base of this algebraic family.

As another example, let X be the A_n-Milnor fibre. Consider $S = \mathrm{Conf}_{n+1}^{\mathrm{unord}}(\mathbb{C})$ as a subset of the space of polynomials of degree n by identifying a configuration c with the polynomial $p_c(z) = \prod_{m \in c}(z - m)$. Take χ to be the family of affine varieties

$$\{x^2 + y^2 + p_c(z) = 0\}$$

parametrised by $c \in S$. Theorem A implies that the induced maps

$$\pi_i(S) \to \pi_i(B\mathrm{Symp}(X))$$

are isomorphisms for $i > 0$. The statement about symplectic mapping class groups is still slightly out of reach as remarked in section 1.3.1.

Remark 1.3.6. *If there is a coarse moduli space $\mathcal{M}(X, L)$ containing the polarised variety (X, L) then a family $(\chi, \mathfrak{L}) \to S$ as above also gives rise to a map $S \to \mathcal{M}(X, L)$. This begs the question: when do the maps*

$$\pi_i(S) \to \pi_i(BSymp(X, \omega))$$
$$\pi_i(S) \to \pi_i(\mathcal{M}(X, L))$$

have the same kernel? It would be very interesting to produce examples where the kernels differ. Since we fixed a Hermitian metric on $\mathfrak{L}$ we actually obtain Kähler metrics on the fibres of the family over S. Assuming that there is a coarse moduli space $\mathcal{K}(X)$ of Kähler metrics on X we get a map $S \to \mathcal{K}(X)$ and a related set of questions about the kernel of the induced map on homotopy.

Stability conditions

One motivation for proving the result on symplectomorphism groups of Milnor fibres was a theorem of Richard Thomas [40]. In [8], Bridgeland introduced a geometrical invariant of a triangulated category $\mathcal{D}$, the space of stability conditions $\mathrm{Stab}(\mathcal{D})$ which possesses an isometric action of the group of autoequivalences of $\mathcal{D}$. To an exact symplectic manifold with vanishing first Chern class one can associate a triangulated category, the derived Fukaya category, by considering the exact Lagrangian submanifolds and their Floer theoretic intersections. The group of symplectic mapping classes which preserve a certain grading then acts by autoequivalences on the derived Fukaya category and hence by isometries on the space of stability conditions. A natural question is then if there is a relationship between the space of stability conditions on the derived Fukaya category and the universal cover of the classifying space of the symplectomorphism group.

The first suggestion that this might be true was the calculation by Richard Thomas [40] of $\mathrm{Stab}(D^{\pi}(\mathrm{Fuk}(A_n))$ where A_n denotes the A_n-Milnor fibre. This is a contractible space with an action of the braid group Br_{n+1}. Indeed this thesis shows that the universal cover of classifying space of the (compactly-supported) symplectomorphism group is contractible.

Forgetting the higher homotopy groups, the space of stability conditions is still relevant for studying the symplectic mapping class group since it provides a space on which the latter group acts. An example of how this technology might applied is the K3 surface. It follows from Seidel's proof of homological mirror symmetry for the quartic K3 surface [36] that the derived Fukaya category of the K3 is equivalent to the derived category of coherent sheaves on the mirror K3. The space of stability conditions on this triangulated category was studied by Bridgeland in [9]. These results suggest that the symplectic mapping class group of the quartic K3 surface is not finitely generated.

1.3.7 Lagrangian spheres in $\mathbb{D}_n$, $n > 4$

For higher blow-ups similar arguments work to disjoin Lagrangians from divisors, but it is hard to find divisors whose complements are as well-understood as T^*S^2. For example, a Lagrangian sphere in the homology class $E_1 - E_2$ in $\mathbb{D}_n$ ($n < 8$) can be disjoined from the exceptional spheres $E_3, \ldots, E_n$. Blowing-down these $n-2$ exceptional spheres leaves us with $\mathbb{D}_2$. If L and L' are Lagrangian spheres in the homology class $E_1 - E_2$ in $\mathbb{D}_n$ ($n < 8$) then we can isotope them both into the complement of the exceptional spheres $E_3, \ldots, E_n$ and blow-down these spheres to points $e_3, \ldots, e_n$. A corollary of theorem E is that any two Lagrangian spheres in $\mathbb{D}_2$ are smoothly isotopic and we can choose that smooth isotopy to avoid the $n-2$ points $e_3, \ldots, e_n$. The result is a smooth isotopy of L and L' in $\mathbb{D}_n$. This is certainly a symplectic phenomenon, as one can always knot smoothly embedded spheres topologically (for example by connect-summing locally with a smoothly knotted $S^2 \subset \mathbb{R}^4$).

1.3.8 Homotopy type

Note that the weak homotopy equivalences proved in the theorems actually suffices to prove homotopy equivalence since, by [28] remark 9.5.6, each of these topological groups has the homotopy type of a countable CW complex.

1.4 Outline of thesis

Part I concerns exclusively Stein surfaces.

Sections 2.1 and 2.2 review the properties of non-compact symplectic manifolds and their (compactly supported) symplectomorphism groups relevant for later arguments. The aim of this section is proving proposition 2.2.4, which shows that if two (possible non-complete) symplectic manifolds arise from different plurisubharmonic functions on the same complex manifold then their compactly supported symplectomorphism groups are weakly homotopy equivalent. This will be useful later as we will be able to identify the symplectomorphism groups of complements of ample divisors by identifying the complement up to *biholomorphism*. Section 2.3 reviews some basic topology needed in the later computations.

Section 3.1 performs the computation of $\pi_k(\mathrm{Symp}_c(T^*\mathbb{RP}^2))$ and can actually be read independently of the rest of the paper. It closely follows [33].

In section 3.2 we prove theorem C which shows that the symplectomorphism group of $\mathbb{C}^* \times \mathbb{C}$ is contractible. The only real difficulty is in proving weak contractibility of a space of symplectic spheres (playing the role of the divisor mentioned above). This proof is postponed to the end of the section and one crucial lemma is proved in appendix A.1. In section 3.3 we prove theorem A on the symplectomorphism groups of the Milnor fibres. Many of the proofs are similar to the case $\mathbb{C}^* \times \mathbb{C}$, so details are sketched noting where technicalities arise.

Part II concerns exclusively Del Pezzo surfaces.

Section 4.1 reviews basic facts on their cohomology. Section 4.2 contains a proof of theorem E assuming the difficult result theorem 9. Section 4.3 describes some moduli spaces of pseudoholomorphic curves which are relevant for the proof of theorem 9.

Chapter 5 is a proof of theorem 9. It begins in section 5.1 with an outline proof which will be expanded upon in section 5.6. Section 5.2 recalls the required background from symplectic field theory. The details of a transversality argument we need are postponed until appendix A.2. Section 5.3 explains how SFT can be applied in our setting. Sections 5.4 and 5.5 complete the analysis which is then put together into a proof in section 5.6.

We finish in chapter 6 by calculating the homotopy type of the symplectomorphism groups of $\mathbb{D}_3$, $\mathbb{D}_4$ and $\mathbb{D}_5$.

1.5 (Un)originality disclaimer

The following sections of the thesis are expositions of standard material for the convenience of the reader: sections 2.3, 4.1, 5.2 and appendix B.

Sections 2.1, 2.2 and 4.3 are probably well-known to experts, but the particular results required for this thesis were not in the literature. The proof of theorem B in section 3.1 is a very simple modification of Seidel's argument in [33]. The proofs in section A.2 were simple translations of [28], chapter 3, into the analytic framework of SFT and were included mostly for completeness and the benefit of the author.

The arguments from sections 3.2, 3.3 and chapter 6 are heavily inspired by the papers of Seidel [37], Abreu [1] and Lalonde-Pinsonnault [25]. Since submitting the results of chapter 6 to arXiv, the author became aware that similar results were obtained independently by Pinsonnault [30].

The ideas from chapter 5 owe much to those of Richard Hind [19].

Part I

Some Stein surfaces

Chapter 2

Preliminaries

2.1 Stein manifolds and Liouville flows

Definition 2.1.1 (Plurisubharmonic functions, Stein manifolds)**.** *A function $\phi : W \to [0, \infty)$ on a complex manifold (W, J) is called* plurisubharmonic *if it is proper and if $\omega = -d(d\phi \circ J)$ is a positive $(1,1)$-form. A triple (W, J, ϕ) consisting of a complex manifold (W, J) and a plurisubharmonic function ϕ on (W, J) is called a* Stein manifold*. A sublevel set $\phi^{-1}[0, c]$ of a Stein manifold (W, J, ϕ) is called a Stein domain.*

Given a Stein manifold (W, J, ϕ) one can associate to it:

- a 1-form $\theta = -d\phi \circ J$ which is a primitive for the positive $(1,1)$-form ω,
- a vector field Z which is ω-dual to θ.

We call θ the Liouville form and Z the Liouville vector field. Let us abstract the notions of Liouville form and vector field further:

Definition 2.1.2. *A* Liouville manifold *is a triple (W, θ, ϕ) consisting of a smooth manifold W, a (Liouville) 1-form θ on W and a proper function $\phi : W \to [0, \infty)$ such that*

- *$\omega = d\theta$ is a symplectic form,*
- *there is a monotonic sequence $c_i \to \infty$ such that on the level sets $\phi^{-1}(c_i)$ the Liouville vector field Z which is ω-dual to θ, satisfies $d\phi(Z) > 0$ everywhere on $\phi^{-1}(c_i)$.*

(W, θ, ϕ) is said to have finite-type *if there is a $k > 0$ such that $d\phi(Z) > 0$ at x for any $x \in \phi^{-1}(k, \infty)$. It is said to be* complete *if the Liouville vector field is complete. A closed sublevel set $\phi^{-1}[0, k]$ of a Liouville manifold is called a* Liouville domain.

Clearly a Stein manifold inherits the structure of a Liouville manifold and a Stein domain inherits the structure of a Liouville domain.

On a Liouville domain, the negative-time Liouville flow always exists and one can use it to define an embedding $\mathrm{Col} : (-\infty, 0] \times \partial W \to W$ with $\mathrm{Col}^*\theta = e^r\theta|_{\partial W}$ and $\mathrm{Col}_*\partial_r = Z$. We can therefore form the *symplectic completion* of (W, θ, ϕ), which is a Liouville manifold:

Definition 2.1.3 (Symplectic completion). *Let (W, θ, ϕ) be a Liouville domain with boundary M and define $\alpha = \theta|_M$. The* symplectic completion $(\hat{W}, \hat{\theta})$ *of (W, θ) is the manifold $\hat{W} = W \cup_{Col} (-\infty, \infty) \times M$ equipped with the 1-form $\hat{\theta}|_W = \theta$, $\hat{\theta}|_{(-\infty,\infty)\times M} = e^r\alpha$. There is an associated symplectic form $\hat{\omega} = d\hat{\theta}$ and Liouville field $\hat{Z}|_W = Z$, $\hat{Z}|_{(-\infty,\infty)\times W} = Col_*\partial_r$ whose flow exists for all times. We may take $\hat{\phi}$ to be any smooth extension of ϕ which agrees with r outside some compact subset.*

There is also a standard construction given a (possibly incomplete) finite-type Stein manifold (W, J, ϕ) to obtain a complete finite-type Stein manifold. Let h denote the function $h(x) = e^x - 1$.

Lemma 2.1.4 ([5], lemma 3.1 and [38] lemma 6). *Define $\phi_h := h \circ \phi$. (W, J, ϕ_h) is a complete Stein manifold of finite-type.*

In fact, ϕ and ϕ_h have the same critical points. If θ and Z are the Liouville form and vector field on (W, J, ϕ) then we write θ_h and Z_h for the corresponding data on (W, J, ϕ_h). This construction is related to the symplectic completion of a sublevel Stein domain of ϕ:

Lemma 2.1.5. *Let k be such that $\phi^{-1}[0, k)$ contains all the critical points of ϕ and set $W_k = \phi^{-1}[0, k]$, which inherits the structure of a Liouville domain. Then $(\hat{W}_k, d\hat{\theta})$ is symplectomorphic to $(W, d\theta_h)$.*

Proof. Pick some $k' > k$. Let h_0 be a function such that:

- $h_0(x) = h(x)$ if $x > k'$,
- $h_0(x) = x$ if $x \leq k$,
- $h_0'(x) > 0$, $h_0''(x) > 0$ for all $x \in [0, \infty)$.

Define $h_t = th + (1 - t)h_0$. Now consider the functions $\phi_t = h_t \circ \phi$ on W for $t \in [0, 1]$. By lemma 6 of [38] they are all plurisubharmonic functions making (W, J, ϕ_t) into a complete Stein manifold and they all have the same set of critical points. Let θ_t denote $d^c\phi_t$. By lemma 5 of [38] there is a smooth family of diffeomorphisms $f_t : W \to W$ such that $f_t^*\theta_t = \theta_0 + dR_t$ for some compactly-supported function $R : W \times [0, 1] \to \mathbb{R}$ with $R_t(x) = R(x, t)$.

By construction, $\phi_1 = \phi_h$ and $\phi_0|_{W_k} = \phi|_{W_k}$. Therefore $\iota = f_1|_{W_k}$ is a symplectic embedding of W_k into $(W, d\theta_h)$ such that $\iota^*\theta_h = \theta|_{W_k} + dR_1$ and there are no critical points of ϕ_h in the complement $W \setminus \iota(W_k)$.

We want to extend ι to a symplectomorphism $\tilde{\iota} : (\hat{W}_k, d\hat{\theta} \to (W, d\theta_h)$. Begin by replacing $\hat{\theta}$ by $\hat{\theta}_R = \hat{\theta} + dR_1$. Of course $d\hat{\theta} = d\hat{\theta}_R$ still as dR_1 is closed. Let Z_R be the $d\hat{\theta}$-dual Liouville field to $\hat{\theta}_R$. Then, on $\iota(W_k)$ $\iota_* Z_R = Z_h$. If Φ^t_R and Φ^t_h denote the time t Liouville flows on $(\hat{W}_k, \hat{\theta}_R)$ and (W, θ_h) respectively then

$$\tilde{\iota} = \Phi^t_h \circ \iota \circ \Phi^{-t}_R : \hat{W}_k \to W$$

defines the required symplectomorphism. Since we have used Φ^t_R and $\Phi^{-t}_R(x) = \Phi^{-t}_h(x)$ for $x \in W_k$ (by equality of Liouville vector fields), $\tilde{\iota}|_{W_k} = \iota$. □

We note another lemma for convenience.

Lemma 2.1.6. *If ϕ_1 and ϕ_2 are complete plurisubharmonic functions on (W, J) with finitely many critical points then $(W, -dd^c\phi_1)$ is symplectomorphic to $(W, -dd^c\phi_2)$.*

Proof. This follows from applying lemma 5 from [38] to the family $t\phi_1 + (1-t)\phi_2$: the associated 2-forms are all compatible with J and hence all symplectic. □

2.2 Symplectomorphism groups

Definition 2.2.1. *Let (W, ω) be a non-compact symplectic manifold and let $\mathcal{K}$ be the set of compact subsets of W. For each $K \in \mathcal{K}$ let $Symp_K(W)$ denote the group of symplectomorphisms of W supported in K, with the topology of $\mathcal{C}^\infty$-convergence. The group $Symp_c(W, \omega)$ of compactly-supported symplectomorphisms of (W, ω) is topologised as the direct limit of these groups under inclusions.*

One important fact we will tacitly and repeatedly use in the sequel is:

Lemma 2.2.2. *Let C be compact and $f : C \to Symp_c(W, \omega)$ be continuous. Then the image of f is contained in $Symp_K(W, \omega)$ for some K.*

Let us restrict attention to the class of Liouville manifolds (W, θ, ϕ) which are of finite-type and satisfy:

Condition 1. *There is a $T \in [0, \infty]$, a $K \in [0, \infty)$ and an exhausting function $f : [0, T) \to [K, \infty)$ such that:*

- *the Liouville flow of any point on $\phi^{-1}(K)$ is defined until time T,*

- *for all $t \in [0, T)$, the Liouville flow defines a diffeomorphism $\phi^{-1}(K) \to \phi^{-1}(f(t))$.*

Examples of such Liouville manifolds include Stein manifolds (where the Liouville flow is the gradient flow of ϕ with respect to the Kähler metric) and symplectic completions of Liouville domains.

A Liouville manifold (W, θ, ϕ) comes with a canonical family of compact subsets $W_r = \phi^{-1}[0, r]$ with natural inclusion maps $\iota_{r,R} : W_r \hookrightarrow W_R$ for all $r < R$. This family is *exhausting*, that is $\bigcup_{r\in[0,\infty)} W_r = W$. When (W, θ, ϕ) satisfies condition 1 we can use this family to better understand the homotopy type of the symplectomorphism group $\mathrm{Symp}_c(W, d\theta)$.

Proposition 2.2.3. *If (W, θ, ϕ) is a Liouville manifold satisfying condition 1 and $K \in [0, \infty)$ is such that $W_K = \phi^{-1}[0, K)$ contains all critical points of ϕ then $Symp_c(W_K, d\theta|_{W_K})$ is weakly homotopy equivalent to $Symp_c(W, d\theta)$.*

Proof. Let Φ^t denote the time t Liouville flow of Z where that is defined. Let $L_{R-r} : \mathrm{Symp}_{W_R}(W, d\theta) \to \mathrm{Symp}_{W_r}(W, d\theta)$ be the map:

$$L_{R-r}(\psi) = \Phi^{r-R} \circ \psi \circ \Phi^{R-r}$$

The inclusions $\iota_{r,R}$ induce inclusions

$$\iota^S_{r,R} : \mathrm{Symp}_{W_r}(W, d\theta) \hookrightarrow \mathrm{Symp}_{W_R}(W, d\theta)$$

for $r, R \geq K$. The maps L_{R-r} are homotopy inverses for $\iota^S_{r,R}$. Similarly, the inclusion

$$\iota^S_c : \mathrm{Symp}_c(W_K, d\theta|_{W_K}) \to \mathrm{Symp}_{W_K}(W, d\theta)$$

has L_ϵ (for any small ϵ) as a homotopy inverse.

We wish to prove that the inclusion

$$\iota : \mathrm{Symp}_c(W_K, d\theta|_{W_K}) \to \mathrm{Symp}_c(W, d\theta)$$

is a weak homotopy equivalence. To see that the maps

$$\iota_\star : \pi_n(\mathrm{Symp}_c(W_K, d\theta|_{W_K})) \to \pi_n(\mathrm{Symp}_c(W, d\theta))$$

are surjective, it suffices to note that the image of any $f : S^n \to \mathrm{Symp}_c(W, d\theta)$ lands in $\mathrm{Symp}_{W_R}(W, d\theta)$ for some R and that $\iota_{0,R} \circ \iota_c$ is a homotopy equivalence. To see injectivity, any homotopy h between maps $f_1, f_2 : S^n \to \mathrm{Symp}_c(W_K, d\theta|_{W_K})$ through maps $S^n \to \mathrm{Symp}_c(W, d\theta)$ lands in some $\mathrm{Symp}_{W_R}(W, d\theta)$ and $\iota_{0,R} \circ \iota_c$ is a homotopy equivalence so h is homotopic to a homotopy $f_1 \simeq f_2$ in $\mathrm{Symp}_c(W_K, d\theta|_{W_K})$. □

Proposition 2.2.4. *If (W, J) is a complex manifold with two finite-type Stein structures ϕ_1 and ϕ_2 then $Symp_c(W, -dd^c\phi_1)$ and $Symp_c(W, -dd^c\phi_2)$ are weakly homotopy equivalent.*

Proof. Let $\omega_i = -dd^c\phi_i$. By proposition 2.2.3, $\mathrm{Symp}_c(W, \omega_i)$ is weakly homotopy equivalent to $\mathrm{Symp}_c(W_{K_i}, \omega_i|_{W_{K_i}})$ for some K_i. Again by proposition 2.2.3, this is weakly homotopy equivalent to $\mathrm{Symp}_c(\hat{W}_{K_i}, \hat{\omega}_i)$. Now by lemma 2.1.5, this is isomorphic to $\mathrm{Symp}_c(W, -dd^c\,(\phi_i)_h)$. But by lemma 2.1.6, $(W, -dd^c\,(\phi_1)_h)$ is symplectomorphic to $(W, -dd^c\,(\phi_2)_h)$ (since $(\phi_i)_h$ is complete). This proves the proposition. □

2.3 Groups associated to configurations of spheres

This section reviews the topology of some well-known (Fréchet-) Lie groups which will crop up frequently later.

2.3.1 Gauge transformations

Let $C \subset X$ be an embedded symplectic 2-sphere in a symplectic 4-manifold and fix a set $\{x_1, \ldots, x_k\}$ of distinct points of C. The normal bundle $\nu = TC^{\omega\perp}$ is a $SL(2,\mathbb{R})$-bundle over C. We will be interested in the group $\mathcal{G}_k$ of symplectic gauge transformations of ν which equal the identity at $x_1, \ldots, x_k$. This will arise when we consider symplectomorphisms of X which fix C pointwise and are required to equal the identity at the k specified points on C.

We first observe that $SL(2,\mathbb{R})$ deformation retracts to the subgroup $U(1)$ which is homeomorphic to a circle. The map

$$\mathrm{ev}_{x_1} : \mathcal{G}_0 \to SL(2,\mathbb{R})$$

taking a gauge transformation to its value at the point x_1 is a fibration whose fibre is $\mathcal{G}_1$. By definition, $\mathcal{G}_1$ is the space of based maps $S^2 \to SL(2,\mathbb{R})$ so

$$\pi_i(\mathcal{G}_1) = \pi_0(\mathrm{Map}(S^{2+i}, SL(2,\mathbb{R}))) = \pi_0(\mathrm{Map}(S^{2+i}, S^1)) = H^1(S^{2+i};\mathbb{Z}) = 0$$

and $\mathcal{G}_1$ is weakly contractible. By the long exact sequence of the fibration ev_{x_1} we see that ev_{x_1} is a weak homotopy equivalence.

Now take the map $\mathrm{ev}_{x_i} : \mathcal{G}_i \to SL(2,\mathbb{R})$. This is again a fibration whose fibre is $\mathcal{G}_{i-1}$. By induction using the homotopy long exact sequence of this fibration, $\pi_0(\mathcal{G}_i) = \mathbb{Z}^{i-1}$ for $i \geq 1$ and $\pi_j(\mathcal{G}_i) = 0$ for $j > 0$.

We can identify generators for $\pi_0(\mathcal{G}_k)$ as follows. For each intersection point x_i the map $\mathrm{ev}_{x_i} : \mathcal{G}_k \to SL(2,\mathbb{R})$ yields a map $\mathbb{Z} = \pi_1(SL(2,\mathbb{R})) \to \pi_0(\mathcal{G}_k)$. The symplectic form gives an orientation of $SL(2,\mathbb{R})$ so there is a preferred element $1 \in \mathbb{Z}$. Let $g_C(x_i) \in \mathcal{G}_k$ be the image of this element in $\pi_0(\mathcal{G}_k)$. These are not independent, but they are canonical and they generate.

To summarise:

$$\begin{aligned}
&\mathcal{G}_0 \simeq S^1 \\
&\mathcal{G}_1 \simeq \star \\
&\mathcal{G}_k \simeq \mathbb{Z}^{k-1},\ k > 1
\end{aligned}$$

2.3.2 Surface symplectomorphisms

Once again, let $C \subset X$ be an embedded symplectic 2-sphere and $\{x_1, \ldots, x_k\}$ a set of k distinct points. Let $\mathrm{Symp}(C, \{x_i\})$ denote the group of symplectomorphisms of C fixing the points x_i. Since this group acts N-transitively on points for any N, we may as well write $\mathrm{Symp}(C, k)$. Moser's theorem implies that $\mathrm{Symp}(C, k)$ is a deformation retract of the group $\mathcal{D}_k$ of diffeomorphisms of S^2 fixing k points. Using techniques of Earle and Eells [13] coming from Teichmüller theory, one can prove:

- $\mathcal{D}_1 \simeq S^1$,
- $\mathcal{D}_2 \simeq S^1$,
- $\mathcal{D}_3 \simeq \star$,
- $\mathcal{D}_5 \simeq P\mathrm{Br}(S^2, 5)/\mathbb{Z}/2$, where $P\mathrm{Br}(S^2, 5)$ is the pure braid group for 5-strands on S^2 and the $\mathbb{Z}/2$ is generated by a full twist (see section 6.1).

Here the S^1 in $\mathcal{D}_1$ can be thought of as rotating around the fixed point. $\mathcal{D}_2$ has a map to $S^1 \times S^1$ measuring the angle of rotation of a diffeomorphism about the two fixed points p_1 and p_2. The kernel of this map is the group $\mathcal{D}_2^c$ of compactly supported diffeomorphisms of $S^2 \setminus \{p_1, p_2\}$, which is weakly equivalent to $\mathbb{Z}$ generated by a Dehn twist in a simple closed curve generating the fundamental group of $S^2 \setminus \{p_1, p_2\}$. In the homotopy long exact sequence of the fibration $\mathcal{D}_2^c \to \mathcal{D}_2 \to (S^1)^2$, one gets:

$$1 \to \pi_1(\mathcal{D}_2) \to \mathbb{Z}^2 \to \mathbb{Z} \to \pi_0(\mathcal{D}_2) \to 1$$

The Dehn twist is the image of $(1, 0)$ and $(0, 1)$ under the map $\mathbb{Z}^2 \to \mathbb{Z}$, so $\mathcal{D}_2 \simeq S^1$. If p_1 and p_2 are antipodal, a 2π rotation around the axis through them gives a non-trivial loop.

2.3.3 Configurations of spheres

We will frequently meet the situation where there is a configuration of embedded symplectic 2-spheres $C = C_1 \cup \cdots \cup C_n \subset X$ in a 4-manifold. Write I for the set of intersection points amongst the components. Suppose that there are no triple intersections amongst components and that all intersections are transverse. Let

- $\mathrm{Stab}(C)$ denote the group of symplectomorphisms of X fixing the components of C pointwise.
- k_i denote the number of intersection points in $I \cap C_i$ and $\mathrm{Symp}(C_i, k_i)$ denote the group of symplectomorphisms of components of C fixing all the intersection points. Write $\mathrm{Symp}(C)$ for the product $\prod_{i=1}^n \mathrm{Symp}(C_i, k_i)$.
- $\mathcal{G}(C_i)$ denote the group of gauge transformations of the normal bundle to $C_i \subset X$ which equal the identity at the k_i intersection points (so $\mathcal{G}(C_i) \cong \mathcal{G}_{k_i}$). Write $\mathcal{G}$ for the product $\prod_{i=1}^n \mathcal{G}(C_i)$.

Let us write $\mathrm{Stab}^0(C)$ for the kernel of the map

$$\begin{aligned} \mathrm{Stab}(C) &\to \mathrm{Symp}(C) \\ \phi &\mapsto (\phi|_{C_1}, \ldots, \phi|_{C_n}) \end{aligned}$$

We will need to understand the homotopy long exact sequence of the fibration

$$\begin{array}{ccc} \mathrm{Stab}^0(C) & \longrightarrow & \mathrm{Stab}(C) \\ & & \downarrow \\ & & \mathrm{Symp}(C) \end{array} \qquad (*)$$

namely

$$\cdots \to \pi_1(\mathrm{Symp}(C)) \to \pi_0(\mathrm{Stab}^0(C)) \to \pi_0(\mathrm{Stab}(C)) \to \cdots$$

There is a map $\mathrm{Stab}^0(C) \to \mathcal{G}$ which sends a symplectomorphism ϕ to the induced map on the normal bundles of components of C. We can understand the composite

$$\pi_1(\mathrm{Symp}(C)) \to \pi_0(\mathrm{Stab}^0(C)) \to \pi_0(\mathcal{G})$$

by thinking purely locally in a neighbourhood of C.

To see this, note that $\pi_1(\mathrm{Symp}(C))$ is only non-trivial if C contains components with $n_i = 1$ or 2 marked points. Suppose $n_i = 1$. There is a Hamiltonian circle action which rotates C_i around the marked point, giving a generating loop γ in $\pi_1(\mathrm{Symp}(C_i, n_i))$.

Pull this back to a Hamiltonian circle action on the normal bundle of C_i in X. By the symplectic neighbourhood theorem this is a local model for X near C_i, so by cutting off the Hamiltonian at some radius in the normal bundle one obtains a path ϕ_t in $\mathrm{Symp}(X)$ consisting of symplectomorphisms which are supported in a neighbourhood of C_i (see figure). The symplectomorphism ϕ_t can be assumed to fix C as a set (indeed it just rotates the component C_i) and $\phi_{2\pi}$ fixes C pointwise, i.e. $\phi_{2\pi} \in \mathrm{Stab}^0(C)$. By definition, $\phi_{2\pi}$ represents the image of $\gamma \in \pi_1(\mathrm{Symp}(C))$ under the boundary map $\pi_1(\mathrm{Symp}(C)) \to \pi_0(\mathrm{Stab}^0(C))$ of the long exact sequence above. A similar story holds when $n_i = 2$. The following lemma is immediate from the definitions.

Lemma 2.3.4. *The image of $\phi_{2\pi}$ under the map $\pi_0(Stab^0(C)) \to \pi_0(\mathcal{G})$ is*

- *($n_i = 1$:) $g_{C_j}(x) \in \pi_0(\mathcal{G}(C_j))$ where $x \in C_i \cap C_j$,*
- *($n_i = 2$:) $g_x(1)g_y(1) \in \pi_0(\mathcal{G}(C_j)) \times \pi_0(\mathcal{G}(C_k))$ where $x \in C_i \cap C_j$ and $y \in C_i \cap C_k$.*

2.3.5 Banyaga's isotopy extension theorem

We will make constant recourse to the following theorem of Banyaga:

Theorem 5 (Banyaga's isotopy extension theorem, see [29], p. 98)**.** *Let (X,ω) be a compact symplectic manifold and $C \subset X$ be a compact subset. Let $\phi_t : U \to X$ be a symplectic isotopy of an open neighbourhood U of C and assume that*

$$H^2(X, C; \mathbb{R}) = 0$$

Then there is a neighbourhood $\mathcal{N} \subset U$ of C and a symplectic isotopy $\psi_t : X \to X$ such that $\psi_t|_{\mathcal{N}} = \phi_t|_{\mathcal{N}}$ for all t.

The proof uses a standard Moser-type argument.

Chapter 3

Computations

3.1 $T^*\mathbb{RP}^2$

Theorem 5. *The group of compactly supported symplectomorphisms of $T^*\mathbb{RP}^2$ with its canonical symplectic form is weakly homotopy equivalent to $\mathbb{Z}$.*

Proof. The proof is an adaptation of Seidel's proof [33] for T^*S^2. The 2-1 cover $S^2 \to \mathbb{RP}^2$ differentiates to a 2-1 cover of tangent bundles. Identifying tangent and cotangent bundles via the round metric, we get a 2-1 cover $T^*S^2 \to T^*\mathbb{RP}^2$. Restrict this to a 2-1 cover of the unit disc subbundles $U^*(S^2) \subset T^*S^2$ and $U^*(\mathbb{RP}^2) \subset T^*\mathbb{RP}^2$:

$$\pi : U^*(S^2) \to U^*(\mathbb{RP}^2)$$

π intertwines the cogeodesic flows on these cotangent disc bundles. Since the cogeodesic flows are periodic, one can symplectically cut along them and π extends to a branched cover of the compactifications:

$$S^2 \times S^2 \to \mathbb{CP}^2$$

branched along the diagonal $\Delta \subset S^2 \times S^2$ (which maps one-to-one onto a conic C in $\mathbb{CP}^2$). Here Δ and C are the reduced loci of the symplectic cut.

Let $\iota : S^2 \times S^2 \to S^2 \times S^2$ be the involution which swaps the two S^2 factors. This is the deck transformation of the branched cover. Let:

- $\mathcal{S}_0^\iota$ denote the group of ι-equivariant symplectomorphisms of $S^2 \times S^2$ which fix Δ pointwise and act trivially on the homology of $S^2 \times S^2$,

- $\mathcal{S}_1^\iota \subset \mathcal{S}_0^\iota$ denote the subgroup of symplectomorphisms compactly supported in the complement of Δ,

- $\mathcal{S}$ denote the group of symplectomorphisms of $\mathbb{CP}^2$ compactly supported in the complement of C. This is homotopy equivalent to the group we are interested in.

An element $\tilde{\phi} \in \mathcal{S}_1^\iota$ descends to an element $\phi \in \mathcal{S}$. $\tilde{\phi}$ is the only ι-equivariant symplectomorphism of $S^2 \times S^2$ which acts trivially on homology and descends to ϕ, so the correspondence $\tilde{\phi} \to \phi$ is a homeomorphism $\mathcal{S}_1^\iota \to \mathcal{S}$. We have therefore reduced the proposition to computing the weak homotopy type of $\mathcal{S}_1^\iota$.

Let $\mathcal{G}$ be the group of gauge transformations of the symplectic normal bundle to $\Delta \subset S^2 \times S^2$. By a standard argument [37] this is homotopy equivalent to S^1. Let $\mathcal{S}_0^\iota \to \mathcal{G}$ be the map taking an ι-equivariant symplectomorphism fixing Δ to its derivative along the normal bundle of Δ. This is a fibration whose fibre is weakly homotopy equivalent to $\mathcal{S}_1^\iota$. The proposition will follow from the long exact sequence of this fibration if we can show that $\mathcal{S}_0^\iota$ is contractible. This is where Gromov's theorem comes in.

Let J_0 denote the product complex structure on $S^2 \times S^2$. Let $\mathcal{J}_0^\iota$ denote the space of ω-compatible almost complex structures J on $S^2 \times S^2$ such that:

- $J \circ \iota_* = \iota_* \circ J$,
- J restricted to $T\Delta$ is equal to J_0.

Lemma 3.1.1. *$\mathcal{J}_0^\iota$ is contractible.*

Proof. Recall that if g is a metric on a manifold X and ω a symplectic form then the endomorphism $A \in \mathrm{End}(TX)$ defined by

$$\omega(\cdot, A\cdot) = g(\cdot, \cdot)$$

satisfies $A^\dagger = -A$, so $A^\dagger A$ is symmetric and positive definite. Let $PA^\dagger P^{-1} = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ be the diagonalisation of $A^\dagger A$. Define the square root $\sqrt{A^\dagger A}$ to be the matrix

$$P^{-1}\mathrm{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n})P.$$

Then $J_A = \sqrt{A^\dagger A}^{-1} A$ is an almost complex structure on X compatible with ω.

Let J be an almost complex structure on $S^2 \times S^2$ compatible with the product form ω. This defines a metric g_0 via $g_0(\cdot,\cdot) = \omega(\cdot, J\cdot)$. Similarly, $\iota^* J$ is an ω-compatible almost complex structure which defines a metric g_1. The space of metrics is convex and the metric $g_t = (1-t)g_0 + tg_1$ defines a family of endomorphisms A_t as above interpolating between J and $\iota^* J$. Now J_{A_t} is a family of almost complex structures interpolating between J and $\iota^* J$ and $J_{A_{1/2}}$ is ι_*-invariant. Hence the contractible space of all ω-compatible almost complex structures (equal to J_0 along Δ) deformation retracts onto the space of ι_*-equivariant ones. $\square$

We now prove that $\mathcal{S}_0^\iota$ is contractible. There is a map $A : \mathcal{S}_0^\iota \to \mathcal{J}_0^\iota$ which sends ϕ to $\phi_* J_0$. The following argument constructs a left inverse B for A.

Gromov's theorem gives J-holomorphic foliations $\mathcal{F}_1^J$ and $\mathcal{F}_2^J$ for each $J \in \mathcal{J}_0^\iota$. In fact the two foliations must be conjugate under ι since $\iota\mathcal{F}_1^J$ is certainly a foliation by J-holomorphic spheres in the homology class $\mathcal{F}_2^J$ and postivity of intersections implies it is the unique one. We also know that, for any $J \in \mathcal{J}_0^\iota$, Δ is J-holomorphic so each leaf of $\mathcal{F}_i^J$ intersects Δ in a single point. Define a (manifestly ι-equivariant) self-diffeomorphism ψ_J of $S^2 \times S^2$ by sending a point p to the pair $(\Lambda_1(p) \cap \Delta, \Lambda_2(p) \cap \Delta)$ where $\Lambda_i(p)$ denotes the unique leaf of $\mathcal{F}_i^J$ passing through p. ψ_J conjugates the pair $(\mathcal{F}_1^J, \mathcal{F}_2^J)$ with the standard pair $(\mathcal{F}_1^{J_0}, \mathcal{F}_2^{J_0})$. Since these standard foliations are ω-orthogonal, it is not hard to see that $\psi_J^*\omega$ is J-tame. Define $\omega_t = t\psi_J^*\omega + (1-t)\omega$, so that

$$\dot{\omega}_t = \psi_J^*\omega - \omega$$

Since ψ_J acts trivially on homology this expression is exact and there exists σ such that $d\sigma = \psi_J^*\omega - \omega$. If X_t is ω_t-dual to σ then the time t flow of X_t defines a sequence of diffeomorphisms ψ_t such that $\psi_0 = \mathrm{id}$ and $\psi_t^*\omega_t = \omega$. Therefore $\psi_{J,t} = \psi_J \circ \psi_t$ is an isotopy from ψ_J to a symplectomorphism $\psi_{J,1}$ of ω. We want $\psi_{J,t}$ to have the following properties:

- $\psi_{J,t}$ is ι-equivariant for all t,
- $\psi_{J,t}$ fixes Δ for all t,
- if $J = \phi_* J_0$ then $\psi_{J,1} = \phi$.

These can be achieved by suitable choice of σ. The first two follow if we require σ to be ι^*-invariant and to vanish when restricted to $TX|_\Delta$. These properties will be achieved momentarily (in a manner such that σ varies smoothly with J). The third property follows if we can take $\sigma \equiv 0$ whenever $d\sigma \equiv 0$; for then if $J = \phi_* J_0$, the foliations $\mathcal{F}_i^J$ are just $\phi(\mathcal{F}_i^{J_0})$ so $\psi_J = \phi$ and $\dot{\omega}_t = 0 = d\sigma$. Taking $\sigma = 0$ yields $\psi_{J,1} = \psi_{J,0} = \phi$.

Fix a metric g and let σ' be the unique g-coexact 1-form for which $d\sigma' = \dot{\omega}_t$ (which exists and varies smoothly with J by Hodge theory). In order to make sure $\psi_{J,t}$ fixes Δ, we want σ to vanish on the bundle $TX|_\Delta$. Since $\psi_J|_\Delta = \mathrm{id}$, $d\sigma' = 0$ along Δ. In particular we may choose a harmonic function f on Δ with $df = \sigma'|_\Delta$ (as $H^1(\Delta; \mathbb{R}) = 0$). Fix an ϵ such that the radius-ϵ subbundle U_ϵ of the normal bundle $T\Delta^\perp$ such that $\exp_g : U_\epsilon \to X$ is injective. Define

$$\tilde{f}(\exp(q, v)) = f(\exp(q, 0)) + \sigma'_{\exp(q,0)}(\exp_* v).$$

This satisfies $d\tilde{f} = \sigma'$ when restricted to $TX|_\Delta$. Fix a cut-off function η which equals 1 inside the radius-$\epsilon/2$ subbundle $U_{\epsilon/2}$ and equals zero near the boundary of U_ϵ. The function $F = \eta\tilde{f}$ is now globally defined and still satisfies $dF = \sigma'$ when restricted to $TX|_\Delta$. The 1-form

$$\sigma := \frac{(\sigma' - dF) + \iota^*(\sigma' - dF)}{2}$$

is now ι^*-invariant, vanishes along Δ and satisfies $d\sigma = \dot{\omega}_t$.

We finally note that σ varies smoothly with J and that the unique coexact σ with $d\sigma = 0$ is $\sigma = 0$, so the third property holds (by Hodge theory).

This proves that $\mathcal{S}_0^\iota$ is homotopy equivalent to $\mathcal{J}_0^\iota$ and hence contractible by lemma 3.1.1. The proposition now follows from the homotopy long exact sequence of the fibration $\mathcal{S}_0^\iota \to \mathcal{G}$, whose fibre is weakly equivalent to $\mathcal{S}_1^\iota$ by the symplectic neighbourhood theorem. □

Remark 3.1.2. *Since all geodesics of $\mathbb{RP}^2$ are closed one can define a Dehn twist in a Lagrangian $\mathbb{RP}^2$ just as one does for S^2. This is actually the generator of the symplectic mapping class group $\mathbb{Z}$, see [33].*

3.2 $\mathbb{C}^* \times \mathbb{C}$

In this section we prove the following theorem:

Theorem 6. *If $\mathbb{C}^* \times \mathbb{C}$ is equipped with the standard (product) symplectic form then $Symp_c(\mathbb{C}^* \times \mathbb{C})$ is weakly contractible.*

3.2.1 Proof of theorem C

We define three holomorphic spheres in $X = \mathbb{CP}^1 \times \mathbb{CP}^1$:

$$C_1 = \{0\} \times \mathbb{CP}^1$$
$$C_2 = \{\infty\} \times \mathbb{CP}^1$$
$$C_3 = \mathbb{CP}^1 \times \{\infty\}.$$

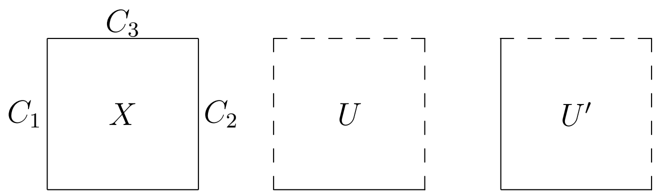
Consider the complements

$$U = X \setminus C_1 \cup C_2 \cup C_3 \qquad\qquad U' = X \setminus C_2 \cup C_3.$$

These are biholomorphic to $\mathbb{C}^* \times \mathbb{C}$ and $\mathbb{C}^2$ respectively. The split symplectic form ω on X restricts to symplectic forms $\omega|_U$ and $\omega|_{U'}$ which are both Stein for the standard (product) complex structures. By proposition 2.2.4,

$$\mathrm{Symp}_c(U) \simeq \mathrm{Symp}_c(\mathbb{C}^* \times \mathbb{C}) \qquad\qquad \mathrm{Symp}_c(U') \simeq \mathrm{Symp}_c(\mathbb{C}^2)$$

and by a theorem of Gromov [16], $\mathrm{Symp}_c(\mathbb{C}^2) \simeq \star$.



We will now define a space on which $\mathrm{Symp}_c(U')$ acts. Let $\mathcal{J}$ denote the space of ω-compatible almost complex structures on X.

Definition 3.2.2. *A* standard configuration *in X is an embedded symplectic sphere S satisfying the following properties:*

- *S is homologous to C_1,*
- *S is disjoint from C_2,*
- *there exists a $J \in \mathcal{J}$ making S, C_2 and C_3 simultaneously J-holomorphic,*
- *there exists a neighbourhood ν of C_3 such that $\nu \cap S$ is equal to $\nu \cap C_1$.*

Let $\mathcal{C}_0$ denote the space of standard configurations, topologised as a quotient

$$\mathit{Map}(\mathbb{CP}^1, X)/\mathit{Diff}(\mathbb{CP}^1)$$

where $\mathit{Map}(\mathbb{CP}^1, X)$ and $\mathit{Diff}(\mathbb{CP}^1)$ are given the $\mathcal{C}^\infty$-topology.

The first important fact about this space is the following proposition whose proof is postponed to section 3.2.6:

Proposition 3.2.3. *$\mathcal{C}_0$ is weakly contractible.*

Given this proposition, we observe that

Lemma 3.2.4. *$\mathit{Symp}_c(U')$ acts transitively on $\mathcal{C}_0$.*

Proof. Since $\mathcal{C}_0$ is weakly contractible it is path-connected. If S_0 and S_1 are two standard configurations let S_t be a path connecting them. Then

$$T_t = S_t \cup C_2 \cup C_3$$

is an isotopy of configurations of symplectic spheres. Since S_t is a standard configuration for each t the isotopy T_t extends to an isotopy of a neighbourhood of T_t by the symplectic neighbourhood theorem. We may assume this isotopy fixes a neighbourhood of $C_2 \cup C_3$ pointwise. Since $H^2(X, T_t; \mathbb{R}) = 0$, Banyaga's symplectic isotopy extension theorem (theorem 5) implies that this isotopy extends to a path ψ_t in $\mathrm{Symp}(X)$. By construction ψ_t is the identity on a neighbourhood of $C_2 \cup C_3$. Hence $\psi_1 \in \mathrm{Symp}_c(U')$ sends S_0 to S_1, proving transitivity. □

The map $\mathrm{Symp}_c(U') \to \mathcal{C}_0 = \mathrm{Orb}(C_1)$ is a fibration by the orbit-stabiliser theorem. We identify the stabiliser:

Lemma 3.2.5. *The stabiliser $Stab(C_1)$ of C_1 under this action is weakly homotopy equivalent to $Symp_c(U)$.*

Proof. Since a symplectomorphism $\phi \in \mathrm{Symp}_c(U')$ fixes a neighbourhood of C_3, an element $\phi \in \mathrm{Stab}(C_1) \subset \mathrm{Symp}_c(U')$ will induce a symplectomorphism $\bar{\phi}$ of C_1 which is compactly supported away from the point $\infty = ([0:1],[1:0]) \in C_1 \cap C_3$. The map $\phi \mapsto \bar{\phi}$ gives a fibration

$$\begin{array}{ccc} \mathrm{Stab}^0(C_1) & \longrightarrow & \mathrm{Stab}(C_1) \\ & & \downarrow \\ & & \mathrm{Symp}_c(C_1, \infty) \end{array}$$

where $\mathrm{Stab}^0(C_1)$ is the group of symplectomorphisms $\phi \in \mathrm{Symp}_c(U')$ which fix C_1 pointwise and $\mathrm{Symp}_c(C_1, \infty)$ denotes the group of compactly supported symplectomorphisms of $C_1 \setminus \{\infty\}$. This latter group is contractible, so it suffices to prove that $\mathrm{Stab}^0(C_1)$ is weakly homotopy equivalent to $\mathrm{Symp}_c(U)$.

Define $\mathcal{G}$ to be the group of symplectic gauge transformations of the normal bundle to C_1 in X which equal the identity on some neighbourhood of $\infty \in C_1$. It follows from section 2.3.1 that this group is weakly contractible. The map $\mathrm{Stab}^0(C_1) \to \mathcal{G}$, which takes a compactly supported symplectomorphism of U' fixing C_1 to its derivative on the normal bundle to C_1, defines a fibration. Weak contractibility of $\mathcal{G}$ implies that the kernel, $\mathrm{Stab}^1(C_1)$, of this fibration is weakly homotopy equivalent to $\mathrm{Stab}^0(C_1)$.

This kernel is the group of compactly supported symplectomorphisms of U' which fix C_1 and act trivially on its normal bundle. By the symplectic neighbourhood theorem, this is weakly homotopy equivalent to the group of compactly supported symplectomorphisms of $U = U' \setminus C_1$. □

Putting all this together, weak contractibility of $\mathrm{Symp}_c(U)$ follows from the long exact sequence of the fibration

$$\begin{tikzcd} \mathrm{Symp}_c(U) \simeq \mathrm{Stab}(C_1) \arrow[r] & \mathrm{Symp}_c(U') \simeq \star \arrow[d] \\ & \mathcal{C}_0 \simeq \star \end{tikzcd} \tag{**}$$

We noted earlier that $\mathrm{Symp}_c(U) \simeq \mathrm{Symp}_c(\mathbb{C}^* \times \mathbb{C})$, so this proves theorem C.

3.2.6 Proof of proposition 3.2.3

To prove weak contractibility of the space of standard configurations, we will need some preliminary work.

Gompf isotopy

We first introduce another space of configurations of spheres.

Definition 3.2.7. *A* nonstandard configuration *in X is an embedded symplectic sphere S satisfying the following properties:*

- *S is homologous to C_1,*
- *S is disjoint from C_2,*
- *S intersects C_3 transversely once at $([0:1],[1:0])$,*
- *there exists a $J \in \mathcal{J}$ making S, C_2 and C_3 simultaneously J-holomorphic,*

Let $\mathcal{C}$ denote the space of standard configurations, topologised as a quotient

$$Map(\mathbb{CP}^1, X)/Diff(\mathbb{CP}^1)$$

where $Map(\mathbb{CP}^1, X)$ and $Diff(\mathbb{CP}^1)$ are given the $\mathcal{C}^\infty$-topology.

These are just like standard configurations but without the requirement that they intersect C_3 in a standard way. An additional subtlety arises from requiring there exists a J making S and C_3 simultaneously holomorphic: it is not even true that two transverse symplectic linear subspaces of a symplectic vector space can be made simultaneously J-holomorphic. This is discussed further below.

There is an inclusion $\iota : \mathcal{C}_0 \hookrightarrow \mathcal{C}$.

Lemma 3.2.8. *ι is a weak homotopy equivalence.*

The proof of this uses a construction due to Gompf. Specifically, the proof of lemma 2.3 from [15] implies the following:

Lemma 3.2.9. *Let N be a symplectic 2-manifold and $E \to N$ be a (disc-subbundle of a) rank-2 vector bundle with a symplectic form ω on the total space making the zero-section both symplectic and symplectically orthogonal to the fibres. Let M be a 2-dimensional symplectic submanifold of E, closed as a subset of E, intersecting N transversely at a single point x. Then there is a symplectic isotopy $M(t)$ of M for which $M(0) = M$, $M(t) = M$ outside a small neighbourhood of $M(t) \cap N$ and $M(t)$ agrees with the fibre E_x in a small neighbourhood of x.*

The isotopy defined in that proof depends on: a choice of neighbourhood of N in which M intersects N once transversely and three auxiliary real parameters. Other choices (such as a cut-off function μ) can be made to depend only on these real parameters. It is clear from the definitions of these parameters that they can be chosen to depend continuously on M as M varies in the space of symplectic submanifolds with the $\mathcal{C}^\infty$-topology, and the same neighbourhood of N can be used for M_1 and M_2 which are close in the $\mathcal{C}^\infty$-topology. We will call the isotopy defined in that proof a *Gompf isotopy*.

It also follows from Gompf's construction that Gompf isotopy preserves the space $\mathcal{C}$. The only subtlety is that $\mathcal{C}$ consists of configurations whose components can be made simultaneously J-holomorphic. One must check that at the intersection points, the tangent planes of the intersecting components can be made simultaneously J-holomorphic. In a Darboux chart ($\cong \mathbb{R}^2 \times \mathbb{R}^2$) around the intersection point, where one of the intersecting components is mapped to the $\mathbb{R}^2 \times \{\star\}$ coordinate plane, transversality of the intersection implies one can write the other component as a graph of a map $f : \mathbb{R}^2 \to \mathbb{R}^2$. The condition that this graph is symplectic is just that $\det(f_\star) > -1$. The condition that the graph can be made simultaneously J-holomorphic with $\mathbb{R}^2 \times \{\star\}$ is that $\det(f_\star) \geq 0$. In the local model used to define it, Gompf isotopy changes $\det(f_\star)$ monotonically towards 0, so it preserves $\mathcal{C}$.

Proof of lemma 3.2.8. Let $f_1,\ f_2 : (S^n, \star) \to (\mathcal{C}_0, C)$ represent homotopy classes $[f_1]$, $[f_2]$ in $\pi_n(\mathcal{C}_0, C)$. We first show that $\iota_\star[f_1] = \iota_\star[f_2]$ only if $[f_1] = [f_2]$. Suppose $S : (S^n \times [1, 2], \star) \to (\mathcal{C}, C)$ is a homotopy between $\iota \circ f_1$ and $\iota \circ f_2$.

By the symplectic neighbourhood theorem there is a neighbourhood ν of C_3, isomorphic to a disc-subbundle of $C_3 \times \mathbb{C}$ and such that $S(x, t) \cap \nu$ satisfies the hypotheses of lemma 3.2.9. Pick the parameters small enough to define a Gompf isotopy Γ for all spheres $S(x, t)$. Then

$$\Gamma : (S^n \times [1, 2] \times [1, 2], \star) \to (\mathcal{C}, C)$$

gives a homotopy from S to $S' = \Gamma(\cdot, \cdot, 2)$. This new homotopy lies entirely in $\mathcal{C}_0$. Since the original maps $f_i : S^n \to \mathcal{C}_0$ landed in the space of standard configurations and Gompf isotopy preserves the property of being a standard configuration, S' is indeed a homotopy connecting f_1 and f_2 in $\mathcal{C}_0$. This proves injectivity of $\iota_\star$.

To prove surjectivity, let $f : (S^n, \star) \to (\mathcal{C}, C)$ represent a homotopy class $[f]$. For some neighbourhood ν of C_3 and sufficiently small choice of parameters one obtains a Gompf isotopy rel C of the image of f into $\mathcal{C}_0$. This gives a homotopy class $[f']$ in $\pi_n(\mathcal{C}_0, C)$ and the Gompf isotopy is a homotopy between $\iota_\star[f']$ and $[f]$.

Since $\iota_\star$ is an isomorphism on homotopy groups, ι is a weak homotopy equivalence. $\square$

Structures making a configuration holomorphic

Let $S = \bigcup_{i=1}^n S_i$ be a union of embedded symplectic 2-spheres in a symplectic 4-manifold X. Suppose that the various components intersect transversely and that there are no triple intersections. Suppose further that there is an ω-compatible J for which all the components S_i are J-holomorphic. Let $\mathcal{H}(S)$ denote the space of ω-compatible almost complex structures J for which all components of S are simultaneously J-holomorphic.

Lemma 3.2.10. *If the components of S intersect one another symplectically orthogonally then the space $\mathcal{H}(S)$ is weakly contractible.*

This lemma is proved in appendix A.1.

Gromov's theory of pseudoholomorphic curves

The following theorem follows from Gromov's theory of pseudoholomorphic curves in symplectic 4-manifolds.

Theorem 6 ([16]). *Given an ω-compatible almost complex structure J on $X = \mathbb{CP}^1 \times \mathbb{CP}^1$ there is a unique J-holomorphic curve through $([0:1],[1:0])$ in the homology class $[C_1]$.*

If we restrict to the space $\mathcal{H}(C')$ of almost complex structures which make C_2 and C_3 into J-holomorphic spheres then this gives us a map

$$G : \mathcal{H}(C') \to \mathcal{C}$$

since by positivity of intersections, the unique J-holomorphic curve through $([0:1],[1:0])$ in the homology class C_1 is then a nonstandard configuration, i.e. it cannot intersect C_2 and it must intersect C_3 transversely once.

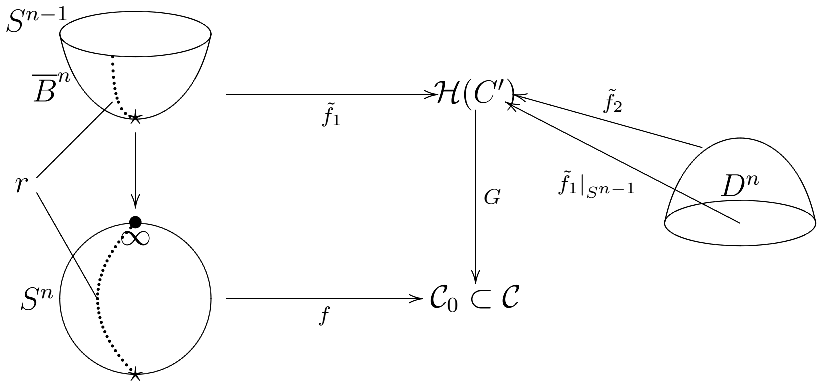
Proof of proposition 3.2.3

Let $f : (S^n, \star) \to (\mathcal{C}_0, C)$ be a based map. We must show that f is nullhomotopic. Let $\infty \in S^n$ be the antipode to $\star$ and let R be the space of great half-circles $r(t)$ connecting $r(0) = \star$ to $r(1) = \infty$. Each such half-circle defines an isotopy $f(r(t))$ of standard configurations. This isotopy extends to a small neighbourhood of the standard configurations by the symplectic neighbourhood theorem and then to a global isotopy $\psi_{r,t} \in \mathrm{Symp}(C)$ by Banyaga's isotopy extension theorem (since $H^2(X, S; \mathbb{R}) = 0$ for all the standard configurations). We can assume this isotopy fixes a neighbourhood of C_2 and C_3 pointwise.

Suppose J_0 is an ω-compatible almost complex structure for which the basepoint $C \in \mathcal{C}_0$ is J_0-holomorphic. Then the standard configuration $f(r(t))$ is $(\psi_{r,t})_\star J_0$-holomorphic. Let $\overline{B}^n = S^n \setminus \{\infty\} \cup S^{n-1}$ be the compactification of $S^n \setminus \{\infty\}$ which adds an endpoint to every open half-circle $\{r(t)\}_{t\in[0,1]}$ (this is the oriented real blow-up of S^n at ∞). Define a map:

$$\tilde{f}_1 : \overline{B}^n \to \mathcal{H}(C')$$
$$\tilde{f}_1(r,t) = (\psi_{r,t})_\star J_0 \in \mathcal{H}(C')$$

which lifts $\overline{B}^n \to S^n \xrightarrow{f} \mathcal{C}_0$ (where the first map collapses S^{n-1} to the point ∞).



The image of the restriction $\tilde{f}_1|_{S^{n-1}} : S^{n-1} \to \mathcal{H}(C')$ consists of ω-compatible almost complex structures which make the standard configuration $f(\infty)$ holomorphic. The space of such almost complex structures ($\mathcal{H}(f(\infty)) \subset \mathcal{H}(C')$) is contractible by lemma 3.2.10 and hence there is a map $\tilde{f}_2 : D^n \to \mathcal{H}(C')$ with $\tilde{f}_2|_{\partial D^n} = \tilde{f}|_{S^{n-1}}$ (where D^n is the closed n-ball) such that $G \circ \tilde{f}_2(x) = f(\infty)$. Glue $\tilde{f}_1$ and $\tilde{f}_2$ to obtain a map $\tilde{f} = \tilde{f}_1 \cup \tilde{f}_2 : S^n \to \mathcal{H}(C')$ for which $G \circ \tilde{f}$ is homotopic to f.

By lemma 3.2.10 $\mathcal{H}(C')$ is contractible, $\tilde{f}$ is nullhomotopic via a map $H : D^{n+1} \to \mathcal{H}(C')$. Therefore $G \circ H$ is a nullhomotopy of f in $\mathcal{C}$. But we have observed (lemma 3.2.8) that $\mathcal{C}_0 \to \mathcal{C}$ is a weak homotopy equivalence, hence f is nullhomotopic in $\mathcal{C}_0$.

3.3 A_n-Milnor fibres

In this section we prove the following theorem:

Theorem 4. *Let W be the A_n-Milnor fibre, the affine variety given by the equation*

$$x^2 + y^2 + z^n = 1,$$

and let ω be the Kähler form on W induced from the ambient $\mathbb{C}^3$. Then the group of compactly supported symplectomorphisms of (W, ω) is weakly homotopy equivalent to its group of components. This group of components injects homomorphically into the braid group Br_n of n-strands on the disc.

3.3.1 Compactification

We begin by compactifying W to a projective rational surface. Let $\xi_k = \exp(2\pi i/k)$ and denote by $P_k(c)$ the polynomial $\frac{c^n-1}{c-\xi_k}$. Define

- X to be the blow-up of $\mathbb{CP}^2$ at the points $\{[\xi_k : 0 : 1]\}_{k=1}^n$. This can be thought of as the subvariety of $M = \mathbb{CP}^2 \times \prod_{k=1}^n \mathbb{CP}^1_k$ given by the equations

  $$a_k y = b_k(x - \xi_k z) \text{ for } k = 1, \ldots, n$$

  in coordinates

  $$([x : y : z], [a_1 : b_1], \ldots, [a_n : b_n]) \text{ on } M.$$

- the pencil of curves P_t which are proper transforms of the pencil of lines through $[0 : 1 : 0]$, parametrised by $t = [x : z]$. In particular, notice that

  $$P_\infty = \{([x : y : 0], [x : y], \ldots, [x : y])\}$$

- the curve $C_{n+1} = P_\infty$.

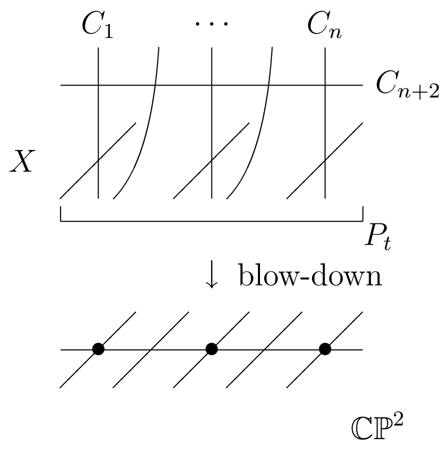
- the curve $C_{n+2} = \{[x : 0 : z]\} \times \prod_{k=1}^n \{[1 : 0]\}$ which is the proper transform of the line through the blow-up points. This is a section of the pencil P_t.

- the exceptional spheres C_k $(1 \leq k \leq n)$ of the blow-up. C_k is given in equations by

  $$\{[\xi_k : 0 : 1]\} \times \{[1 : 0]\} \times \ldots \times \mathbb{CP}^1_k \times \ldots \times \{[1 : 0]\}$$

  and constitutes one component of the singular curve P_{ξ_k}.

- the divisor $C' = C_{n+1} \cup C_{n+2}$ and its complement $U' = X \setminus C'$.
- the divisor $C = C' \cup \bigcup_{i=1}^{n} C_k$ and its complement $U = X \setminus C$.



Lemma 3.3.2. *U' is biholomorphic to W and U is biholomorphic to $\mathbb{C}^* \times \mathbb{C}$.*

Proof. We proceed by introducing affine charts on U' and writing down explicit partial maps $W \dashrightarrow U'$ which glue compatibly to give a biholomorphism.

Let $\mathbb{C}_+$, $\mathbb{C}_-$ denote the upper and lower (Zariski) hemispheres of $\mathbb{CP}^1$ respectively. For any partition σ of $\{1, \ldots, n\}$ into two sets σ_+ and σ_- let $(\pm)^\sigma(k)$ be ± 1 if $k \in \sigma_\pm$ respectively and consider the affine open set

$$M_\sigma = \mathbb{C}^2 \times \prod_{k=1}^{n} \mathbb{C}_{(\pm)^\sigma(k)} \subset M$$

Denote by U'_σ the intersection $U' \cap M_\sigma$. Notice that unless $|\sigma_-| \leq 1$, y cannot vanish, so

$$\bigcup_{|\sigma_-|\geq 2} U'_\sigma = \bigcap_{|\sigma_-|\geq 2} U'_\sigma = \{((x, y), [x - \xi_1 : y], \ldots, [x - \xi_n : y]) : y \neq 0\}$$

We define σ^k to be the partition with $\sigma_- = \{k\}$. Since we have excised C_{n+2}, we do not need to consider the partition σ^∞ with $\sigma_- = \emptyset$.

Recall that $W = \{(a, b, c) \in \mathbb{C}^3 : a^2 + b^2 + c^n = 1\}$. For each σ we define a partial map $\psi_\sigma : W \dashrightarrow U'_\sigma$ (i.e. a map whose domain is implicitly defined as the region on which the map is well-defined) by

$$\psi_\sigma(a, b, c) = ((c, a + ib), [c - \xi_1 : a + ib], \ldots, [c - \xi_n : a + ib])$$

for $|\sigma_-| \geq 2$ and

$$\psi_{\sigma^k}(a, b, c) = ((c, a + ib), [c - \xi_1 : a + ib], \ldots, [a - ib : P_k(c)], \ldots [c - \xi_n : a + ib])$$

Since $a^2+b^2=(a+ib)(a-ib)=1-c^n$ and $P_k(c)(c-\xi_k)=c^n-1$, the standard $a_i \mapsto 1/a_i$ transition maps for the affine charts on $\mathbb{CP}^1$ allow us to glue these partial maps into a global biholomorphism $W \to U'$.

To identify U and $\mathbb{C}^* \times \mathbb{C}$, notice that U is just the subset

$$\left\{\left((x,y),[x-\xi:y],\ldots,[x-\xi^{n+1}:y]\right): y \neq 0\right\}$$

from above. □

We now construct a symplectic form on the compactification X. Let Ω denote the symplectic form on $M=\mathbb{CP}^2\times\prod_{k=1}^n \mathbb{CP}^1_k$ which is the product of the Fubini-Study forms on each factor, normalised to give a complex line area 1. X inherits a symplectic form ω from its embedding in M and it is easy to see that

$$\begin{aligned}
[C_{n+2}] &= [C_{n+1}] - \sum_{k=1}^{n} [C_k] \\
\omega(C_{n+1}) &= n+1 \\
\omega(C_{n+2}) &= 1 \\
\omega(C_k) &= 1 \\
P.D.[\omega] &= n[C_{n+1}] + [C_{n+2}] \\
c_1(X) &= 3[C_{n+1}] - \sum_{k=1}^{n} [C_k]
\end{aligned}$$

Moreover, the following observation will prove extremely useful:

Lemma 3.3.3. *At the intersection points $C_{n+2}\cap P_\infty$ and $C_{n+2}\cap C_k$, the various intersecting components are all ω-symplectically orthogonal. Indeed C_{n+2} is symplectically orthogonal to all the members of the pencil P_t.*

This can be checked by hand. We also notice that since U is the intersection of an affine open of M with X that $\omega|_U$ arises from a plurisubharmonic function. One can pick multiplicities for the components of C' such that the resulting divisor satisfies the Nakai-Moishezon criterion for ampleness, so by the first proposition in [14], chapter 1, section 2 there is also a plurisubharmonic function $\phi_{C'}$ on U' for which $-dd^c\phi_{C'}=\omega|_{U'}$. By proposition 2.2.4, this implies:

Lemma 3.3.4. *$Symp_c(U) \simeq Symp_c(\mathbb{C}^*\times\mathbb{C})$ and $Symp_c(U') \simeq Symp_c(W)$.*

3.3.5 Proof of theorem 4

We now proceed to the proof of theorem 4. As before we introduce the notion of a standard configuration:

Definition 3.3.6. *A* standard configuration *in X will mean an* unordered *n-tuple of embedded symplectic spheres $\{S_i\}_{i=1}^n$ satisfying the following properties:*

- *each S_i is disjoint from $C' = C_{n+1} \cup C_{n+2}$,*
- *$[S_i] = [C_i]$,*
- *there exists a $J \in \mathcal{J}$ for which all the spheres S_i, C_{n+1} and C_{n+2} are J-holomorphic.*
- *there is a neighbourhood ν of C_{n+2} such that, for all $i \in \{1, \ldots, n\}$,*

$$S_i \cap \nu = P_{t_i} \cap \nu$$

where $t_i = S_i \cap C_{n+2}$.

Let $\mathcal{C}_0$ denote the space of all standard configurations.

Note that we really do want unordered configurations here in order to obtain the full (rather than the pure) braid group.

Proposition 3.3.7. *$\mathcal{C}_0$ is weakly contractible.*

Proof. The proof of this is similar to the proof of proposition 3.2.3. The Gompf isotopy argument is slightly more involved, as there are several components that need to be isotoped at once. The crucial input from Gromov's theory of pseudoholomorphic curves is the following:

Theorem 7 ([28], corollary 3.3.4)**.** *Let S_i be an embedded symplectic 2-sphere in a symplectic 4-manifold with homological self-intersection $S_i \cdot S_i = -1$. Then for any ω-compatible almost complex structure J there is a unique embedded J-holomorphic sphere homologous to S_i.*

□

Let $\mathrm{Conf}(n)$ denote the space of configurations of unordered points in $C_{n+2} \setminus \{\infty\}$, where ∞ denotes the point of intersection between C_{n+2} and P_∞. Define the map

$$\begin{aligned} \Psi : \mathcal{C}_0 &\to \mathrm{Conf}(n) \\ S = \bigcup_{i=1}^{n} S_i &\mapsto \{S_1 \cap C_{n+2}, \ldots, S_n \cap C_{n+2}\}. \end{aligned}$$

Proposition 3.3.8. Ψ *is a fibration.*

Proof. Let Y be a test space and

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d] & \mathcal{C}_0 \arrow[d, "\Psi"] \\ Y \times I \arrow[r, "h"'] & \mathrm{Conf}(n) \end{tikzcd}$$

be a commutative test diagram for the homotopy lifting property which defines a fibration. We must show that h lifts to a map $\tilde{h} : Y \times I \to \mathcal{C}_0$ and which commutes with the other maps.

Let us first notice that h can be arbitrarily closely approximated by maps which are smooth in the I-direction (for example, by theorem 2.6 of [20], chapter 2). So if we can prove homotopy lifting for such maps then by taking limits of approximation sequences we have proved it for all maps. Let us therefore assume that h is smooth in the I-direction.

For clarity of exposition let us also assume that $n = 1$ and explain how the proof should be modified for $n > 1$ at the end.

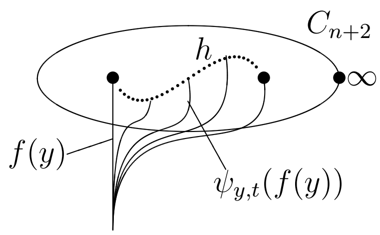
The idea behind constructing $\tilde{h}$ will be to find symplectomorphisms $\psi_{y,t} : X \to X$ such that:

- $\psi_{y,t}$ preserves $C' = C_{n+1} \cup C_{n+2}$ and fixes C_{n+1} pointwise,
- $\psi_{y,t}(S) \in \mathcal{C}_0$ for any $S \in \mathcal{C}_0$,
- $\Psi(\psi_{y,t}(f(y))) = h(y,t)$.

The lift $\tilde{h}$ will then be defined by

$$\tilde{h}(y,t) = \psi_{y,t}(f(y))$$

For example, if $Y = \{y\}$ this is just path-lifting:



Notice that $\Psi(\psi_{y,t}(f(y))) = \psi_{y,t}(\Psi(f(y)))$, so it suffices that $\psi_{y,t}$ restricted to C_{n+2} satisfies $\psi_{y,t}(\Psi(f(y))) = h(y,t)$. In fact, we start by constructing the restriction $\bar{\psi}_{y,t}$ of $\psi_{y,t}$ to C_{n+2}.

First, pick a continuous family of parametrised Darboux discs $\{B_p \xrightarrow{\iota_p} C_{n+2} \setminus \{\infty\}\}$, one centred at each point $p \in C_{n+2} \setminus \{\infty\}$. For each $(y,t) \in Y \times I$ let $X(y,t) = \frac{\partial h(y,t)}{\partial t}$ and let $v(y,t)$ be the pullback of this vector to $B_{h(y,t)}$ along $\iota_{h(y,t)}$. Extend v to a constant vector field on $B_{h(y,t)}$. This is generated by a linear Hamiltonian function which we can multiply by a cut-off function (equal to 1 on a neighbourhood of $0 \in B_{h(y,t)}$) to obtain a Hamiltonian which is compactly supported in the ball. Pulling back this Hamiltonian along $\iota^{-1}_{h(y,t)}$ gives a Hamiltonian $H_{y,t}$ on C_{n+2}, supported in a neighbourhood of $h(y,t)$. We call the time t flow of this Hamiltonian $\bar{\psi}_{y,t}$. Notice that $\bar{\psi}_{y,t}(h(y,0)) = h(y,t)$.

Let N be a small tubular neighbourhood of C_{n+2} and let η be a radial cut-off function on N which equals 1 on a neighbourhood of C_{n+2}. The Hamiltonian function $\eta H_{y,t}$ generates a time t flow $\psi_{y,t}$ which extends $\bar{\psi}_{y,t}$ in such a way that it fixes C_{n+1} and sends standard configurations to standard configurations.

To prove the result when $n > 1$ the same construction is used in a Darboux chart near each point of $C_{n+2} \cap S$. This means that now we fix a (continuously varying) choice of n disjoint Darboux discs over each configuration of n distinct points in $C_{n+2} \setminus \{\infty\}$, modifying the proof accordingly. □

The space $\mathrm{Conf}(n)$ of configurations of n points in the disc has fundamental group Br_n and it follows from the theory of fibrations that this group acts freely on the set of components of the fibre $\mathcal{F}$ of Ψ. Since $\mathcal{C}_0$ is (weakly) contractible, this action is transitive. Moreover, $\mathrm{Conf}(n)$ is a $K(\mathrm{Br}_n, 1)$-space, so the long exact sequence of homotopy groups associated to Ψ implies:

Lemma 3.3.9. $\mathcal{F} \simeq \pi_0(\mathcal{F})$.

We have an action of $\mathrm{Symp}_c(U')$ on the fibre $\mathcal{F} = \Psi^{-1}(\{\xi_1, \ldots, \xi_n\})$. Let us restrict attention to the orbit $\mathrm{Orb}(E)$ of the standard configuration of exceptional curves $C_1, \ldots, C_n$.

Lemma 3.3.10. *This consists of a union of connected components of $\mathcal{F}$.*

Proof. If $S_0 \in \mathrm{Orb}(E) \subset \mathcal{F}$ and $S_1 \in \mathcal{F}$ lie in the same connected component of $\mathcal{F}$ then they are isotopic through standard configurations; let S_t denote such an isotopy. Consider the isotopy $T_t = S_t \cup C'$ of configurations of symplectic spheres. By the assumption that S_t is a standard configuration for each $t \in [0,1]$, this isotopy extends to an isotopy of neighbourhoods ν_t of T_t. We may also assume that this isotopy fixes a neighbourhood of $D = C_{n+2} \cup P_\infty$. The fact that $H^2(X, T_t; \mathbb{R}) = 0$ implies that this isotopy extends to a global symplectic isotopy of X (by theorem 5). Hence there is a symplectomorphism of X compactly supported on the complement of C' (i.e. an element of $\mathrm{Symp}_c(U')$) which sends S_0 to S_1, so $\mathrm{Orb}(E)$ consists of a union of connected components of $\mathcal{F}$. □

Lemma 3.3.11. *The stabiliser Stab(E) is weakly contractible.*

Proof. Arguing as in lemma 3.2.5, we see that this is equivalent to showing $\mathrm{Symp}_c(U)$ is weakly contractible. But we have already observed that

$$\mathrm{Symp}_c(U) \simeq \mathrm{Symp}_c(\mathbb{C}^* \times \mathbb{C}) \simeq \star.$$

□

The orbit-stabiliser theorem gives a fibration:

$$\begin{tikzcd} \mathrm{Stab}(E) \arrow[r] & \mathrm{Symp}_c(U') \arrow[d] \\ & \mathrm{Orb}(E) \end{tikzcd}$$

Weak contractibility of $\mathrm{Stab}(E)$ and of the components of $\mathrm{Orb}(E)$ implies that

$$\pi_i(\mathrm{Symp}_c(U')) = 0 \text{ for } i > 0.$$

We finally prove that:

Proposition 3.3.12. *$\pi_0(\mathrm{Symp}_c(U'))$ injects homomorphically into the braid group Br_n.*

Proof. The symplectomorphisms $\psi_{y,t}$ used to construct lifts in the proof of the fibration property of Ψ are supported in some arbitrarily small neighbourhood of C_{n+2}. The action of $\mathrm{Symp}_c(U')$ is by definition supported on the complement of a neighbourhood of C_{n+2}. This implies that the action of $\pi_0(\mathrm{Symp}_c(U'))$ on $\pi_0(\mathcal{F})$ commutes with the action of Br_n coming from the fibration Ψ. When restricted to the components of $\mathrm{Orb}(E) \subset \mathcal{F}$ we have seen that the $\pi_0(\mathrm{Symp}_c(U'))$ action is free. The Br_n action is free on the whole of $\pi_0(\mathcal{F})$. The claim that $\pi_0(\mathrm{Symp}_c(U'))$ injects into the braid group now follows from the following elementary lemma:

Lemma 3.3.13. *Let G and H be groups acting on a set A. Suppose H acts freely and transitively on A and that G acts freely on an orbit $\mathrm{Orb}_G(a)$ for some $a \in A$. In particular, for each $b \in A$ there exists a unique h_b such that $h_b(a) = b$. Define a map*

$$f : G \to H$$
$$f(\sigma) = h^{-1}_{\sigma(a)}$$

This map is injective. If the actions commute then it is a homomorphism.

□

Part II

Some rational surfaces

Chapter 4

Symplectic Del Pezzo surfaces

4.1 Cohomology and Lagrangian spheres

4.1.1 Del Pezzo surfaces and symplectic forms

A Del Pezzo surface X is a smooth complex variety whose anticanonical bundle $-K_X$ is *ample*, that is the sections of some tensor power of $-K_X$ define an embedding of X into a projective space. The restriction of the ambient Fubini-Study form is then a Kähler form, ω, on X. Note that we are normalising the Fubini-Study form to give a line in $\mathbb{CP}^N$ area 1. The following classification theorem is well-known (see [31] for example)

Theorem 8. *A Del Pezzo surface is biholomorphic to one of:*

- *A smooth quadric surface $Q \subset \mathbb{CP}^3$ or, equivalently, a product $\mathbb{CP}^1 \times \mathbb{CP}^1$ (thinking of the $\mathbb{CP}^1$ factors as rulings on the quadric surface),*
- *A blow-up, $\mathbb{D}_n$, of $\mathbb{CP}^2$ at n points in general position for $n < 8$.*

For convenience, we recall the second homology groups of these surfaces:

- $H_2(Q, \mathbb{Z}) = \mathbb{Z}^2$ is generated by the classes of the two rulings $\alpha = [\mathbb{CP}^1 \times \{\cdot\}]$ and $\beta = [\{\cdot\} \times \mathbb{CP}^1]$. The intersection pairing is given by $\alpha^2 = 0 = \beta^2$ and $\alpha \cdot \beta = 1$. The first Chern class is Poincaré dual to $2\alpha + 2\beta$.
- $H_2(\mathbb{D}_n, \mathbb{Z}) = \mathbb{Z}^{n+1}$ is generated by the class H of a line in $\mathbb{CP}^2$ and the classes $\{E_i\}_{i=1}^n$ of the n exceptional spheres. The intersection pairing is given by $H^2 = 1$, $E_i \cdot E_j = -\delta_{ij}$ and $H \cdot E_i = 0$. The first Chern class is Poincaré dual to $3H - \sum_{i=1}^n E_i$.

By definition, some multiple of the first Chern class is represented by the Kähler form so these are monotone symplectic manifolds.

Remark 4.1.2. *If $n < 7$ then the anticanonical bundle is* very ample *i.e. its sections already define an embedding into projective space. For such blow-ups of $\mathbb{CP}^2$, the induced Kähler form ω lies in the cohomology class $3H - \sum_{i=1}^{n} E_i$. It is sometimes too inexplicit to be useful, so we also work with a form ω' obtained by performing* symplectic blow-up *(see [29], section 7.1) in n symplectically (and holomorphically) embedded balls of volume $1/2$ in $(\mathbb{CP}^2, 3\omega_{FS})$ centred at n points in general position. Recall that under symplectic blowing up, a ball of volume $\pi^2\lambda^4/2$ is replaced by a symplectic -1-sphere of area $\pi\lambda^2$, so the cohomology class of ω' is again $3H - \sum_{i=1}^{n} E_i$. Since ω' is Kähler for the complex structure of the Del Pezzo surface, by Moser's theorem it is symplectomorphic (indeed isotopic) to ω. For this reason, we will sometimes blur the distinction between ω and ω', writing ω for both.*

For $n \geq 7$, one can always rescale ω so that its cohomology class is $3H - \sum_{i=1}^{n} E_i$. We call this the "anticanonical Kähler form" in the case when the anticanonical bundle is only ample.

4.1.3 Lagrangian spheres

The next lemma describes the homology classes in Del Pezzo surfaces which contain Lagrangian 2-spheres.

Lemma 4.1.4. *If L is a Lagrangian sphere in Q then it represents one of the homology classes $\pm(\alpha - \beta)$. If L is a Lagrangian sphere in $\mathbb{D}_n$ then it either represents a* binary class *of the form $E_i - E_j$, a* ternary class *of the form $\pm(H - E_i - E_j - E_k)$ (if $n \geq 3$) or a* senary class *of the form $\pm(2H - \sum_{k=1}^{6} E_{i_k})$ (if $n \geq 6$).*

Proof. The requirement that L is a Lagrangian sphere means that $[L]^2 = -2$ and $c_1([L]) = 0$. For Q, if $[L] = A\alpha + B\beta$ then $2AB = -2$ and so $A = -B = \pm 1$. For $\mathbb{D}_n$, suppose $[L] = dH + \sum a_i E_i$. Then $c_1([L]) = 0$ implies that

$$3d + \sum a_i = 0$$

which, coupled with $[L]^2 = -2$, gives

$$\left(\frac{-\sum a_i}{3}\right)^2 - \sum a_i^2 = -2$$

or (after some reworking)

$$(9-n)\sum a_i^2 + \sum_{j<k} (a_j - a_k)^2 = 18$$

Since each summand is positive, it is a matter of combinatorics to check the claim. □

Definition 4.1.5. *We call a Lagrangian sphere* binary*,* ternary *or* senary *according to the arity of its homology class as defined in the previous lemma.*

Notice that there is an intrinsic distinction here; it is not merely a matter of the basis we have chosen for $H_2(X, \mathbb{Z})$. For instance, in $\mathbb{D}_3$ there are six binary classes ($\{E_i - E_j\}_{i \neq j}$) and one ternary class ($H - E_1 - E_2 - E_3$). Computing intersection numbers: $(E_i - E_j) \cdot (E_k - E_\ell) = \pm 1$ (when $\{i, j\} \neq \{k, \ell\}$) while $(E_i - E_j) \cdot (H - E_1 - E_2 - E_3) = 0$. The symplectomorphism group acts with two orbits on the Lagrangian classes in $H_2(\mathbb{D}_3, \mathbb{Z})$: a binary one and a ternary one.

4.1.6 Birational relations

Apart from the blow-down maps

$$\rho_n : \mathbb{D}_n \to \mathbb{CP}^2$$

there are also blow-down maps

$$\pi_n^{ij} : \mathbb{D}_n \to Q$$

for $n \geq 2$. To see this, observe that Q is a quadric surface in $\mathbb{CP}^3$ and one can birationally project it from a point $p \in Q$ to a hypersurface $\mathbb{CP}^2 \subset \mathbb{CP}^3$. This map, ϕ, is defined away from p. It collapses the lines α_p and β_p through p to points $a = \alpha_p \cap \mathbb{CP}^2$ and $b = \beta_p \cap \mathbb{CP}^2$, and its image otherwise misses the line E through a and b.

The graph of ϕ inside $Q \times \mathbb{CP}^2$ is therefore isomorphic to $\mathbb{D}_2$:

$$\begin{tikzcd} Q & \mathbb{D}_2 \arrow[l, "\pi_2"'] \arrow[d, "\rho_2"] \\ & \mathbb{CP}^2 \end{tikzcd}$$

If we write π_2 and ρ_2 for the projections of this graph to Q and $\mathbb{CP}^2$ respectively then ρ_2 collapses the two spheres in the graph which respectively project one-to-one onto α_p and β_p in Q. Similarly, π_2 collapses the sphere in the graph which projects via ρ_2 onto E. Thus $\pi_2 : \mathbb{D}_2 \to Q$ is a blow-down. The homology class of the exceptional sphere is $H - E_1 - E_2$.

Let us introduce the notation S_{ij} for the homology class $H - E_i - E_j$ in $H_2(\mathbb{D}_n, \mathbb{Z})$.

The map $\widetilde{\pi}_2$ is defined by blowing up the graph of ϕ:

$$\begin{tikzcd} Q_{n-1} & \mathbb{D}_n \arrow[l, "\widetilde{\pi}_2"'] \arrow[d, "\rho_n"] \\ & \mathbb{CP}^2 \end{tikzcd}$$

where Q_{n-1} indicates Q blown-up in $n-1$ points. Finally π_n^{ij} is defined by composing $\widetilde{\pi}_2$ with the blow-down map to Q. The indices ij are to indicate that the exceptional spheres of π_n^{ij} are taken to be $\{E_k\}_{k\neq i,j}$ and $S_{ij} = H - E_i - E_j$.

Remark 4.1.7. *There is a symplectomorphism from the anticanonical Kähler form on $\mathbb{D}_n$ with the symplectic blow-up of the anticanonical form on Q (compare with remark 4.1.2).*

We construct some Lagrangian spheres in the binary homology class $E_i - E_j$ of $\mathbb{D}_n$:

Definition 4.1.8. *Let $\overline{\Delta}$ be the antidiagonal Lagrangian sphere in $Q = \mathbb{CP}^1 \times \mathbb{CP}^1$, i.e. the graph of the antipodal antisymplectomorphism $\mathbb{CP}^1 \to \mathbb{CP}^1$. If the symplectic balls used to perform the blow-up π_n^{ij} of Q in $n-1$ points are chosen disjoint from $\overline{\Delta}$ then it lifts to a Lagrangian sphere in $(\mathbb{D}_n, \omega')$ and hence specifies a Lagrangian sphere $\widetilde{\Delta}_{ij}$ in $(\mathbb{D}_n, \omega)$ since $\omega \cong \omega'$.*

4.2 Disjointness from divisors: proving theorem E

Theorem E is proved by isotoping Lagrangian spheres until they are disjoint from a set of divisors whose complement is symplectomorphic to a compact subset of T^*S^2. Figures 4.1 and 4.2 describe the relevant systems of divisors. The diagrams are to be interpreted as a union of smooth divisors, one for each line in the diagram, such that the homology class of the divisor corresponding to a given line is the one by which the line is labelled.

Theorem 9. *A binary Lagrangian sphere in the homology class $E_1 - E_2$ in $\mathbb{D}_n$ (for $n \leq 4$) can be isotoped off a configuration of smooth divisors as shown in Figure 4.1. A ternary Lagrangian sphere in the homology class $H - E_1 - E_2 - E_3$ can be isotoped off a configuration of smooth divisors as shown in Figure 4.2.*

Proposition 4.2.1. *In each case the complement U of the divisor contains a unique Lagrangian sphere up to isotopy.*

Proof. In each case we observe that U is biholomorphic to an affine quadric surface:

- **Binary:** In figure 4.1, the pictured divisors are total transforms of the diagonal in $\mathbb{CP}^1 \times \mathbb{CP}^1$ under π_n^{12}.

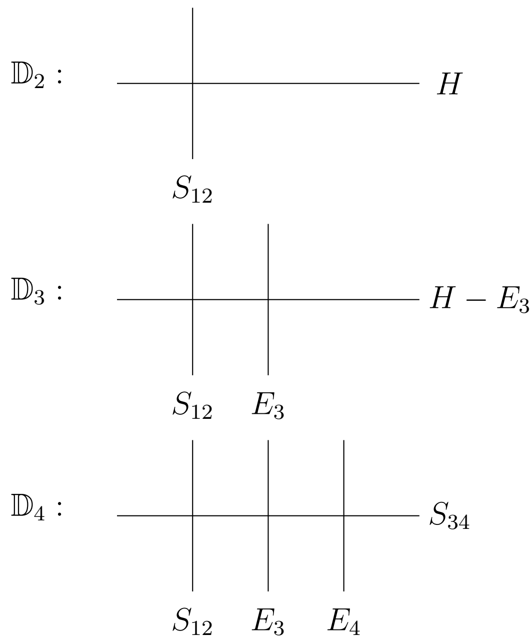


Figure 4.1: The configuration of smooth divisors from which a binary Lagrangian sphere (in the homology class $E_1 - E_2$) can be made disjoint by a Lagrangian isotopy.

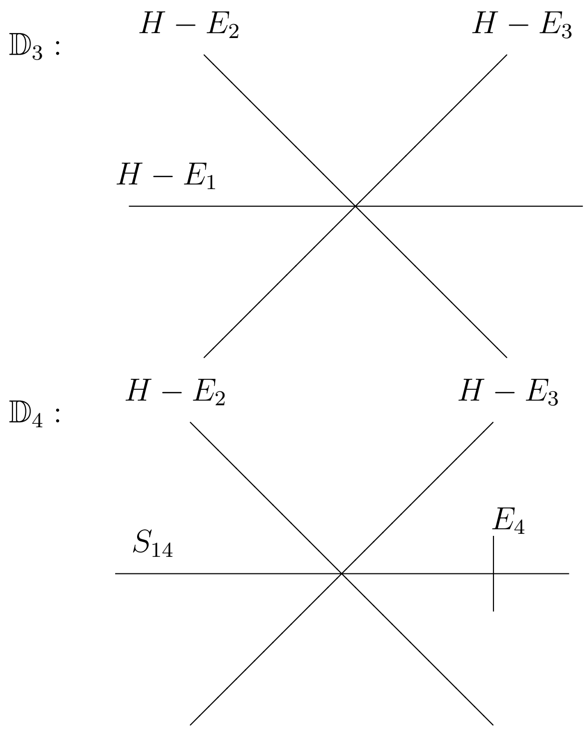


Figure 4.2: The configuration of smooth divisors from which a ternary Lagrangian sphere (in the homology class $H - E_1 - E_2 - E_3$) can be made disjoint by a Lagrangian isotopy.

- **Ternary:** In the $\mathbb{D}_3$ part of figure 4.2, the linear system of the divisor homologous to $H - E_1$ is a pencil with no basepoints and two nodal members (corresponding to the decompositions $S_{12}+E_2$ and $S_{13}+E_3$ of the homology class). The other divisors ($H-E_2$ and $H-E_3$) are sections of this pencil. Excising the pictured divisor gives a Lefschetz fibration over the disc with conic fibres and two nodal singularities. Using this Lefschetz fibration we can establish a biholomorphism with the affine quadric and its standard conic fibration with two singular fibres. For ternary spheres in $\mathbb{D}_4$, the same trick works by looking at the linear system of $H - E_1$ and excising the singular fibre $S_{14} + E_4$ plus two sections (as depicted in the figure).

Now that we have established biholomorphism of U with the affine quadric, notice that in each case we can give multiplicities to the components of the divisor so that it is linearly equivalent to the anticanonical class. Therefore we can assume that the restriction of ω to U is the symplectic form associated to a plurisubharmonic function ϕ on U. It follows from lemma 2.2.4 that the symplectic completion of a sublevel set of ϕ is symplectomorphic to the affine quadric. The affine quadric is symplectomorphic to the total space of T^*S^2.

If there are two Lagrangian spheres L_0 and L_1 in U, they lie inside a sublevel set of ϕ. Since the completion Y of this sublevel set if symplectomorphic to T^*S^2, Hind's theorem gives us an isotopy L_t in Y between L_0 and L_1. Let ψ_t be the negative Liouville flow on Y. For large T, $\psi_{-T}(L_t)$ is a Lagrangian isotopy between $\psi_{-T}(L_0)$ and $\psi_{-T}(L_1)$. Therefore the three-stage isotopy

$$\psi_t(L_0)|_{t=0}^{t=-T},\ \psi_{-T}(L_t)|_{t=0}^{t=1},\ \psi_t(L_1)|_{t=-T}^{t=0}$$

interpolates between L_0 and L_1 whilst remaining inside U. □

Proof of theorem E. Theorem 9 implies that a binary Lagrangian sphere can be isotoped into the complement of a specified divisor. The previous proposition tells us this complement contains a unique Lagrangian sphere up to Lagrangian isotopy in that space. □

4.3 Pseudoholomorphic curves

This chapter makes heavy use of the Gromov-Witten theory of genus 0 pseudoholomorphic curves as explained in [28]. In appendix B we recall the basics of Gromov-Witten theory and the geometric theorems pertinent to dimension 4. We refer the reader to [28] for proofs, where they are excellently presented. The purpose of the next section is to prove that we can see the configurations of divisors specified in theorem 9 and their linear systems even after perturbing the complex structure. Let (X, ω) be a symplectic Del Pezzo surface $\mathbb{D}_n$ with its monotone blow-up form. We will assume $n \leq 7$. With this understood, we omit the target space X from the notation for a moduli space of J-holomorphic maps. J will denote an ω-compatible almost complex structure on $(X, \omega) = \mathbb{D}_n$. The following propositions are proved in sections 4.3.5, 4.3.11 and 4.3.16 respectively.

Proposition 4.3.1. *For any $i \in \{1, \ldots, n\}$ and any ω-compatible J on X there is a unique J-holomorphic stable curve $E_i(J)$ representing the homology class E_i. This curve is smooth, simple and embedded.*

This result is well-known (see for example, [27], lemma 3.1) but we include a proof for completeness.

Proposition 4.3.2. *For any $i \in \{1, \ldots, n\}$ and any ω-compatible J on X,*

$$ev : \overline{\mathcal{M}}_{0,1}(H - E_i, J) \to X$$

is a homeomorphism.

Proposition 4.3.3. *For any ω-compatible J on X,*

$$ev_2 : \overline{\mathcal{M}}_{0,2}(H, J) \to X \times X$$

is surjective.

Let $\Xi(J)$ denote the union of the spheres $E_i(J)$. Let $\mathcal{J}_x$ denote the (non-empty) space of J such that $x \notin \Xi(J)$. This is non-empty because symplectomorphisms act transitively on points and $\mathcal{J}_x$ is non-empty for some x. Consider the evaluation map

$$\mathrm{ev}_1 : \overline{\mathcal{M}}_{0,1}(H, J) \to X$$

and define the space $\overline{\mathcal{M}}_{0,0}(H, x, J) = \mathrm{ev}_1^{-1}(x)$ (which we can think of as unmarked stable J-curves in the class H which pass through x). This final proposition is also proved in section 4.3.16:

Proposition 4.3.4. *Denote by $\mathbb{P}^J_x X$ the space of J-complex lines in $T_x X$. For $J \in \mathcal{J}_x$, the map*

$$\overline{\mathcal{M}}_{0,0}(H, x, J) \to \mathbb{P}^J_x X$$

sending a stable curve through x to its complex tangent at the marked point x is both well-defined (as $x \notin \Xi(J)$) and a homeomorphism.

4.3.5 Area 1 classes: E_i, S_{ij}

Lemma 4.3.6. *A J-holomorphic sphere u in X with area $\int_{S^2} u^*\omega = 1$ is simple and embedded.*

Proof. Since the area is minimal amongst non-zero spherical classes, u cannot factor through a branched cover hence it is simple. Embeddedness will come from the adjunction formula:

$$\delta(u) \leq A \cdot A - \langle c_1(X), A \rangle + 2$$

where $A = u_*[S^2] \in H_2(X, \mathbb{Z})$. Since $\langle c_1(X), A \rangle = E(u) = 1$, it remains to show that $A \cdot A < 0$, for then the adjunction inequality becomes an equality $\delta(u) = 0$, meaning that u is an embedded sphere.

Suppose that $A = \alpha H + \sum_i \beta_i E_i$. Then

$$\begin{aligned} \langle c_1(X), A \rangle &= 3\alpha + \sum_i \beta_i \\ A \cdot A &= \alpha^2 - \sum_i \beta_i^2 \end{aligned}$$

so

$$\begin{aligned} A \cdot A &= \frac{(1 - \sum_i \beta_i)^2}{9} - \sum_i \beta_i^2 \\ 9A \cdot A &= k + 1 - \sum_{i<j} (\beta_i - \beta_j)^2 - \sum_i (\beta_i + 1)^2 - (8 - k) \sum_i \beta_i^2 \end{aligned}$$

We are required to show that this is strictly negative for all possible choices of β_i, which reduces to tedious case analysis. Once β_i is large enough, the term $\sum_i \beta_i^2$ becomes large and negative, so there are very few cases that need to be checked. In order to obtain an integer class A, we also require that $\sum_i \beta_i \equiv 1 \mod 3$, which is useful at a number of points in the case analysis. Note also that the proof fails for the case $n = 8$, due to the existence of the class $3H - E_1 - \cdots - E_8$ with square 1 and area 1. □

Corollary 4.3.7. *For any J, a J-sphere of area 1 is automatically regular.*

Proof. This follows directly from the automatic transversality lemma 20 once we know these spheres are embedded, since $c_1(X)$ evaluates to 1 on these homology classes. □

Corollary 4.3.8. *For any J, the only J-spheres of area 1 lie in the homology classes:*

- E_i, S_{ij} *when* $k < 5$,

- *E_i, S_{ij}, $2H - \sum_{j=1}^{5} E_{i_j}$ when $k \geq 5$.*

Proof. By corollary 4.3.7 such a sphere u is embedded and automatically regular. If u represents a homology class A, then the adjunction inequality reads

$$0 = \delta(u) \leq A \cdot A - c_1(A) + 2$$

so

$$-1 \leq A \cdot A$$

In the proof of lemma 4.3.6 we saw that for such a sphere, $A \cdot A \leq -1$, and the only homology classes with $\langle [\omega], A \rangle = 1$ and $A \cdot A = -1$ are the ones listed in the statement of the corollary. □

Lemma 4.3.9. *Any genus 0 stable curve with one marked point and energy 1 is modelled on a single vertex tree.*

Proof. Certainly the tree for a stable curve of energy 1 can only have one non-constant component as 1 is the minimal energy for a pseudoholomorphic sphere. If it had a ghost bubble then it would have a ghost bubble corresponding to a leaf of the tree. The domain of this component would have at most two special points as there is only one marked point, contradicting stability of the curve. □

Lemma 4.3.10. *For any J there are unique J-holomorphic representatives of the area 1 classes E_i, S_{ij} and (if possible) $2H - \sum_{j=1}^{5} E_{i_j}$.*

Proof. This is clear in the standard almost complex structure. Let E denote one of these area 1 classes. The evaluation map

$$\mathcal{M}^*_{0,1}(E, J_0) \to X$$

is a cycle in the homology class E (the Ω-limit set is empty). By lemma 4.3.9 the set of trees for which we must check GW-regularity is just the one-vertex tree and there are no edges so GW-regularity reduces to usual regularity of the almost complex structure. Therefore any J is GW-regular by corollary 4.3.7. Hence by theorem 17 the bordism class of the evaluation map (and hence the homology class of the image) is independent of J. Thus for any J there is a J-holomorphic sphere in any of these three classes.

Uniqueness follows because these classes have homological self-intersection -1 and J-holomorphic curves intersect positively in dimension 4, so whenever two representatives intersect they must share a component. However, J-holomorphic representatives are smooth by lemma 4.3.6 and therefore have a single component. □

4.3.11 Area 2 classes: $H - E_i$

Lemma 4.3.12. *A smooth J-holomorphic sphere u in the homology class $H - E_i$ is simple and embedded.*

Proof. $H - E_i$ is a primitive class, hence any pseudoholomorphic representative is simple. In this case the adjunction formula gives

$$\begin{aligned}\delta(u) &\leq (H - E_i) \cdot (H - E_i) - \langle c_1(X), H - E_i \rangle + 2 \\ &= 0 - 2 + 2 = 0\end{aligned}$$

and again u must be embedded. □

Corollary 4.3.13. *For any J, a J-sphere in the homology class $H - E_i$ is automatically regular.*

Proof. As before, this follows directly from the automatic transversality lemma 20 once we know these spheres are embedded, since $c_1(X)$ evaluates to 2 on these homology classes. □

In particular, all smooth $H - E_i$-curves in the standard complex structure are regular. Let us examine the corresponding Gromov-Witten pseudocycle in a standard (integrable) complex structure obtained by blowing-up the standard structure on $\mathbb{CP}^2$ at n generic points. The image of the evaluation map:

$$\mathrm{ev} : \mathcal{M}^*_{0,1}(H - E_i, J_0) \to X$$

is a pseudocycle in X with complement the reducible complex codimension 1 subvariety consisting of the $2(n-1)$ exceptional curves in classes E_j and S_{ij} for all $j \neq i$.

Lemma 4.3.14. *For any J, a genus 0 stable curve with one marked point in the moduli space $\overline{\mathcal{M}}_{0,1}(H - E_i, J)$ falls into one of three categories:*

- *A smooth J-sphere with a marked point,*
- *A nodal curve with two smooth components, one marked, each of area 1,*
- *A nodal curve with three components connected according to the tree $\bullet - \dot{\circ} - \bullet$ where the middle vertex corresponds to a marked ghost bubble and the outer spheres are each of area 1.*

Proof. There can be at most two non-constant components because the total area of any stable curve in the homology class $H - E_i$ is 2 and the minimal area of a pseudoholomorphic sphere is 1. There can be at most one ghost bubble because there is only one marked point to stabilise it. $\square$

We have shown that all components of stable curves of the three types shown above are regular and also that the moduli spaces of nodal curves are nonempty. It remains to check transversality of the ev^E map to Δ^E to get GW-regularity.

- For the tree $\bullet$, there are no edges, so GW-regularity reduces to usual regularity.
- For the tree $\bullet\!\!-\!\!\bullet$, lemma B.2.6 reduces edge transversality to transversal intersection of the two energy 1 components (in this case the kernel of $D_{u_\alpha}\overline{\partial}_J$ is trivial as the u_α curves are regular and index 0). Since these components are smooth J-spheres in the homology classes E_j and S_{ij} with $E_j \cdot S_{ij} = 1$, by theorem 18 they intersect once transversely. Hence J is regular for this tree.
- For the tree $\bullet\!\!-\!\!\dot{\circ}\!\!-\!\!\bullet$ with a marked ghost bubble on the middle vertex γ, edge transversality is automatic from lemma B.2.6 since for both nodal points, $\ker D_{u_\gamma}\overline{\partial}_J$ has (real) dimension 4.

The result of this is that any J is GW-regular for the classes $H - E_i$, and theorem 17 implies that the bordism class of the pseudocycle

$$\mathrm{ev} : \mathcal{M}^*_{0,1}(H - E_i, J_0) \to X$$

is independent of the almost complex structure we chose.

Corollary 4.3.15. *For any ω-compatible almost complex structure J, there is a dense set of points in X in the image of*

$$ev : \mathcal{M}^*_{0,1}(H - E_i, J) \to X$$

Proof. By lemma B.2.4 (1) there is a dense set of points which are strongly transverse to the pseudocycle ev. The intersection number of such a point with the image of ev is 1, since that is the case for the standard complex structure and this intersection number is independent of the bordism class of the pseudocycle by lemma B.2.4 (3), which is independent of the almost complex structure by the remarks above. $\square$

Proof of proposition 4.3.2. It suffices to show that the evaluation map is bijective, for then it is a continuous bijection from a compact space $\overline{\mathcal{M}}_{0,1}(H - E_i, J)$ to a Hausdorff space X and hence a homeomorphism.

Surjectivity: From corollary 4.3.15 there is a dense set of points in the image of

$$\mathrm{ev} : \mathcal{M}^*_{0,1}(H - E_i, J) \to X$$

Suppose $x \in X \setminus \mathrm{ev}(\mathcal{M}^*_{0,1}(H - E_i, J))$. Pick a sequence of points

$$x_i \in \mathrm{ev}(\mathcal{M}^*_{0,1}(H - E_i, J))$$

tending to x and a sequence of smooth marked J-curves (u_i, z_i) with $u_i(z_i) = x_i$. This sequence has a Gromov convergent subsequence, by the Gromov compactness theorem, whose limit is a stable J-curve $[u, z]$ in $\overline{\mathcal{M}}_{0,1}(H - E_i, J)$ with $u(z) = x$.

Injectivity: Suppose there were a point $x \in X$ and distinct stable maps $[u, z]$ and $[u', z'] \in \overline{\mathcal{M}}_{0,1}(H - E_i, X)$ for which $u(z) = x = u'(z')$.

If u and u' were both smooth curves then they would have to have the same image, or else they would intersect at x and this intersection would contribute positively to their (zero) homological intersection by McDuff's theorem on positivity of intersections. If they had the same image, they would be reparametrisations of the same smooth curve, and hence correspond to the same stable map in the moduli space.

If u were smooth and u' were nodal then (forgetting marked points) u' would be a stable J-curve corresponding to a splitting

$$H - E_i = S_{ij} + E_j, \; j \neq i$$

of homology classes. This follows directly from lemmas 4.3.14, 4.3.8 and the definition of the forgetful map $\overline{\mathcal{M}}_{0,1}(H - E_i, J) \to \overline{\mathcal{M}}_{0,0}(H - E_i, J)$ (see [28], 5.1.9). In particular, neither component of u' can intersect the image of u without contributing positively to either $(H - E_i) \cdot S_{ij} = 0$ or $(H - E_i) \cdot E_j = 0$. Hence there cannot be a point x in both the image of u and the image of u'.

Finally, consider the case when u and u' were both nodal. If they were stable curves corresponding to different splittings $S_{ij} + E_j$ and $S_{ik} + E_k$ then none of their components could possibly intersect contradicting their both passing through x. If they were to correspond to the same splitting of the homology class then their images would be geometrically indistinct. If the marked point were not mapped to the node then u and u' would clearly be reparametrisations of the same stable marked curve. If the marked point were mapped to the node then both stable maps would be modelled on the tree $\bullet\!-\!\dot{\circ}\!-\!\bullet$ with the marked point on the middle vertex, corresponding to the ghost bubble at the node. Any two such stable maps are equivalent by reparametrising the sphere corresponding to the middle vertex, so again $[u, z] = [u', z']$ as stable maps. $\square$

4.3.16 Area 3 classes: H

As for curves in the class $H - E_i$, it is easily shown that a smooth curve in the class H is simple, embedded and hence automatically regular. We will examine the Gromov-Witten pseudocycle

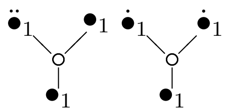
$$\mathrm{ev}_2 : \mathcal{M}^*_{0,2}(H, J) \to X \times X$$

for a standard complex structure obtained by blowing-up n generic points on $\mathbb{CP}^2$. If $(a, b) \in X \times X$ is a pair of distinct points neither of which lies on an exceptional curve E_j then there is a unique line through them lifted from the line in $\mathbb{CP}^2$. If $a = b \notin E_j$ for any j then there is a $\mathbb{CP}^1$ of lines (some of which are singular curves: total transforms of lines through blow-up points) through $a = b$. Therefore the image of this standard pseudocycle has complement the reducible complex codimension 1 subvariety of pairs (a, b) where one or both of a or b lies on a curve E_j.

Lemma 4.3.17. *For any J, a genus 0 stable curve with two marked points in the moduli space $\overline{\mathcal{M}}_{0,2}(H, J)$ falls into one of 22 categories. We show these pictorially below. The symbol $\dot{\bullet}_q$ denotes a non-constant J-sphere with area q and as many marked points as dots. The symbol $\dot{\circ}$ denotes a ghost bubble with as many marked points as dots.*

$$\begin{array}{lll}
\ddot{\bullet}_3 & \ddot{\bullet}_2\text{-}\bullet_1 & \ddot{\bullet}_1\text{-}\bullet_1\text{-}\bullet_1 \\
\bullet_3 - \ddot{\circ} & \dot{\bullet}_2\text{-}\dot{\bullet}_1 & \bullet_1\text{-}\ddot{\bullet}_1\text{-}\bullet_1 \\
 & \bullet_2\text{-}\ddot{\bullet}_1 & \dot{\bullet}_1\text{-}\dot{\bullet}_1\text{-}\bullet_1 \\
 & \bullet_2\text{-}\bullet_1 - \ddot{\circ} & \dot{\bullet}_1\text{-}\bullet_1\text{-}\dot{\bullet}_1 \\
 & \ddot{\circ} - \bullet_2\text{-}\bullet_1 & \dot{\bullet}_1 - \dot{\circ} - \bullet_1\text{-}\bullet_1 \\
 & \bullet_2 - \dot{\circ} - \dot{\bullet}_1 & \bullet_1 - \dot{\circ} - \dot{\bullet}_1\text{-}\bullet_1 \\
 & \dot{\bullet}_2 - \dot{\circ} - \bullet_1 & \bullet_1 - \dot{\circ} - \bullet_1\text{-}\dot{\bullet}_1 \\
 & \bullet_2 - \ddot{\circ} - \bullet_1 & \ddot{\circ} - \bullet_1\text{-}\bullet_1\text{-}\bullet_1 \\
 & \bullet_2 - \dot{\circ} - \dot{\circ} - \bullet_1 & \bullet_1 - \ddot{\circ} - \bullet_1\text{-}\bullet_1 \\
 & & \bullet_1 - \dot{\circ} - \dot{\circ} - \bullet_1\text{-}\bullet_1 \\
 & & \bullet_1 - \dot{\circ} - \bullet_1 - \dot{\circ} - \bullet_1
\end{array}$$

Proof. The only cases that need to be ruled out are:



which would clearly result in three distinct area 1 curves intersecting at a single point. These classes must add up to H, and the only possibility (given corollary 4.3.8) is that the three curves are S_{ij}, E_i and E_j. Since E_i and E_j cannot intersect, these trivalent configurations are ruled out. $\square$

The area 1 components are all understood: by corollary 4.3.8 they are of the form E_j,

S_{ij} or $2H-\sum_{j=1}^{5} E_j$. This latter case cannot occur, since the other components of a stable curve with total class H would have to sum to $-H+\sum_{j=1}^{5} E_j$. This cannot be achieved by two area 1 curves, and there is no area 2 curve in this homology class since it would be somewhere injective (this is a primitive homology class) and have self-intersection -4, contradicting the adjunction inequality

$$0 \leq A \cdot A - c_1(A) + 2 = -4$$

The area 2 components are therefore understood to be either $H-E_j$ or $H-S_{ij} = E_i+E_j$. This latter case is subsumed into the 3-component stable degenerations, as the unique curve in the class E_i+E_j is the (disconnected) union of the exceptional spheres E_i and E_j. One may check that all possible trees are GW-regular (as in the previous section) and so any J is GW-regular for the class H. Theorem 17 implies that the bordism class of the pseudocycle

$$\mathrm{ev}_2 : \mathcal{M}^*_{0,2}(H,J) \to X \times X$$

is independent of J.

Corollary 4.3.18. *For any ω-compatible J there is a dense set $\mathcal{D}$ of points in $X \times X$ in the image of ev_2.*

Proof of proposition 4.3.3. Let $(x,y) \in X \times X$ and pick a sequence of points (x_i,y_i) in $\mathcal{D} \subset X \times X$ (as defined in corollary 4.3.18) tending to (x,y). By the corollary, we can choose a sequence of stable curves $u_i \in \mathrm{ev}_2^{-1}(x_i,y_i)$ and there is a subsequence of these which Gromov converges to a stable curve in $\overline{\mathcal{M}}_{0,2}(H,J)$ through (x,y). This proves surjectivity. □

Proof of proposition 4.3.4. Fix u, a smooth J-curve homologous to H but not passing through x. Consider the set $\upsilon = \mathrm{ev}_2^{-1}(\{x\} \times u)$ of stable curves through x hitting points of u.

Lemma 4.3.19. *$ev_2|_\upsilon : \upsilon \to \{x\} \times u$ is a bijection.*

Surjectivity comes immediately from proposition 4.3.3. Injectivity will follow from positivity of intersections. Two curves homologous to H intersecting at x must intersect transversely as x does not lie on a component $E_i(J)$ by assumption. If they were also to intersect at $p \in u$ they would have intersection number greater than 1 by positivity of intersections, but $H \cdot H = 1$.

Since ev_2 is continuous and all spaces involved are compact and Hausdorff, υ is homeomorphic to the 2-sphere $\{x\} \times u$. We will now show that $\overline{\mathcal{M}}_{0,0}(H,x,J)$ is homeomorphic to the 2-sphere.

There are continuous forgetful maps

$$f_2 : \overline{\mathcal{M}}_{0,2}(H,J) \to \overline{\mathcal{M}}_{0,0}(H,J)$$
$$f_1 : \overline{\mathcal{M}}_{0,1}(H,J) \to \overline{\mathcal{M}}_{0,0}(H,J)$$

The restriction of f_1 to $\overline{\mathcal{M}}_{0,0}(H,x,J) = \mathrm{ev}_1^{-1}(x) \subset \overline{\mathcal{M}}_{0,1}(H,J)$ is a homeomorphism. To see this, first note that $x \notin \Xi(J)$ implies that x is not a node of any stable curve in $\overline{\mathcal{M}}_{0,0}(H,x,J)$. By lemma 4.3.17 and the results of the earlier sections, if $u \in \overline{\mathcal{M}}_{0,0}(H,x,J)$ then all components of u are simple, embedded curves. Furthermore x occurs on precisely one of the components of u. Therefore the restriction of f_1 to $\overline{\mathcal{M}}_{0,0}(H,x,J)$ is a continuous bijection of compact Hausdorff spaces.

The restriction of f_2 to v lands in $\overline{\mathcal{M}}_{0,0}(H,x,J)$ and is a bijection (as every sphere through x will also hit u). By the same point-set topological reasoning it is a homeomorphism. Therefore $\overline{\mathcal{M}}_{0,0}(H,x,J)$ is homeomorphic to a 2-sphere.

Finally, we consider the map

$$\tau : \overline{\mathcal{M}}_{0,0}(H,x,J) \to \mathbb{P}^J_x X$$

which sends a stable curve through x to its complex tangent at x. This is well-defined since by assumption $J \in \mathcal{J}_x$, so $x \notin \Xi(J)$. By positivity of intersections, any two distinct stable curves through x in the homology class H intersect transversely at x, so τ is injective. But a continuous injection from a 2-sphere to a 2-sphere is a homeomorphism, so to prove the proposition it suffices to show τ is continuous.

Lemma 4.3.20. *τ is a continuous map.*

Since the Gromov topology is metrizable it suffices to prove that τ is sequentially continuous, i.e. that the complex tangent at x of a Gromov-limit v of a sequence v_i of stable curves through x is the limit of their complex tangents. Gromov convergence implies $\mathcal{C}^\infty$-convergence of v_i to v on compact subsets away from the nodes and since $x \notin \Xi(J)$ by assumption x is always a smooth point of v_i and of v. Hence the claim follows. □

Chapter 5

Lagrangian spheres: Proof of theorem 9

5.1 Navigation

To help the reader navigate their way through this proof, I begin by sketching the general idea. Take a Lagrangian sphere L in some homology class in one of the Del Pezzos $X = \mathbb{D}_2$, $\mathbb{D}_3$ or $\mathbb{D}_4$ and pick the corresponding configuration C of holomorphic curves as specified in section 4.2. Pick an ω-compatible almost complex structure on X which looks like a certain standard complex structure in a tubular neighbourhood of L (see section 5.3 for specifics) and deform it near the boundary of the tubular neighbourhood so that a longer and longer translationally invariant neck forms. We call this family J_t of almost complex structures a *neck-stretch*. For each J_t there will be a pseudholomorphic configuration of curves C_t homologous to C. As the length of this neck tends to infinity, one can use symplectic field theory (SFT) to analyse what happens to C_t (see sections 5.4 and 5.5 for binary and ternary Lagrangian spheres respectively). The result of this analysis is that for large J_t the configurations must be disjoint from L. The rest of the argument (section 5.6) uses Banyaga's theorem to construct a symplectic isotopy which performs this disjunction.

5.2 Symplectic field theory

5.2.1 The setting

Contact-type hypersurfaces

Definition 5.2.2. *A* contact-type hypersurface *in a symplectic $2n$-manifold (X, ω) is a codimension 1 submanifold M for which there exists a collar neighbourhood $N \cong (-\epsilon, \epsilon) \times M$ and a* Liouville vector field *η defined on N which is transverse to $\{0\} \times M$ and satisfies $\mathcal{L}_\eta \omega = \omega$.*

To a contact-type hypersurface one associates:

- The *contact 1-form* $\lambda = \iota_\eta \omega$,
- The *contact hyperplane distribution* $\zeta = \ker \lambda$,
- The *Reeb vector field* R satisfying $\iota_R d\lambda = 0$, $\lambda(R) = 1$.

Definition 5.2.3. *Given a contact-type hypersurface M, the product $\mathbb{R} \times M$ admits a symplectic structure $d(e^t\lambda)$ and the symplectic manifold $\mathbb{S}(M) = (\mathbb{R} \times M, d(e^t\lambda))$ is called the symplectisation of M. We also write $\mathbb{S}_\pm(M)$ for the positive/negative parts $\mathbb{R}_\pm \times M$ of $\mathbb{S}(M)$.*

Any contact-type hypersurface $M \subset X$ has a neighbourhood which is symplectomorphic to a neighbourhood of $\{0\} \times M$ in $\mathbb{S}(M)$. This symplectomorphism is realised by the flow of the Liouville field (which is to be identified with ∂_t in $\mathbb{R} \times M$).

Definition 5.2.4. *Let M be a contact-type hypersurface in (X, ω) with Liouville vector field η and let J be a complex structure on the bundle $\zeta \to M$. If $d\lambda(\cdot, J\cdot)$ is a nondegenerate bundle metric then we say J is $d\lambda$-compatible. We can define an almost complex structure $\tilde{J}$ on $\mathbb{S}(M)$ by requiring that:*

- *$\tilde{J}$ is $\mathbb{R}$-invariant,*
- *$\tilde{J}\partial_t = R$,*
- *$\tilde{J}|_\zeta = J$.*

If J is $d\lambda$-compatible then $\tilde{J}$ is $d(e^t\lambda)$-compatible. We say that $\tilde{J}$ is a cylindrical almost complex structure *on $\mathbb{S}(M)$. An ω-compatible almost complex structure on X which restricts to a cylindrical almost complex structure on a collar neighbourhood N of M is said to be* adjusted to *M* for the Liouville field *η.*

Contact-type boundary, cylindrical ends

Now let (X, ω) be a compact symplectic manifold with a boundary component M. Suppose that there is:

- a collar neighbourhood $N \cong (-\epsilon, 0] \times M$ of M,
- a Liouville field η defined in $(-\epsilon, 0) \times M$,
- a smooth extension of η to $(-\epsilon, \epsilon') \times M$ (for some ϵ') which is transverse to $\{0\} \times M$.

Then we say that this boundary component of X is of *contact-type*. Furthermore, if η points out of X along M then we say that M is *convex* and we say it is *concave* otherwise. The collar neighbourhood N of a convex (respectively concave) boundary component M is symplectomorphic to a neighbourhood in $\mathbb{S}_-(M)$ (respectively $\mathbb{S}_+(M)$).

The Liouville field gives us all the data we formerly had on a contact-type hypersurface, namely a contact 1-form, a contact hyperplane distribution and a Reeb vector field.

Definition 5.2.5. *The* symplectic completion *of a compact symplectic manifold* (X, ω) *with contact-type boundary* M *comprising* b *components* $\bigcup_{i=1}^b M_i$ *is the union*

$$\overline{X} = X \cup \bigcup_{i=1}^{b} \mathbb{S}_\pm(M_i)$$

where $\pm$ *indicates* $+$ *if* M_i *is convex and* $-$ *if* M_i *is concave. The noncompact parts in the union are called the* ends of $\overline{X}$. *The identifications for convex (respectively concave) ends are via symplectomorphisms defined by the Liouville flows* η_i *on neighbourhoods* $N_i \cong (-\epsilon, 0] \times M_i$ *(respectively* $N_i \cong [0, \epsilon) \times M_i$*) of* M_i.

Thus $\overline{X}$ is a non-compact symplectic manifold and we write $\overline{\omega}$ for its symplectic form. $\overline{X}$ is diffeomorphic to the interior $\mathring{X} \subset X$ via a diffeomorphism ϕ:

- ϕ is the identity on $X \setminus \bigcup_{i=1}^b N_i$,
- ϕ sends each $(-\epsilon, \infty) \times M_i$ (or $(-\infty, \epsilon) \times M_i$) to $\mathring{N}_i \cong (-\epsilon, 0) \times M_i$ (respectively $(0, \epsilon) \times M_i$) by a diffeomorphism of the form $(t, m) \mapsto (g_i(t), m)$,
- the function $g_i : (-\epsilon, \infty) \to (-\epsilon, 0)$ (respectively $g_i : (-\infty, \epsilon) \to (0, \epsilon)$):
  - is monotone and concave,
  - coincides with $t \mapsto -\epsilon e^{-t}/2$ on $(0, \infty)$ (respectively $t \mapsto \epsilon e^t/2$ on $(-\infty, 0)$)
  - coincides with the identity near $-\epsilon$ (respectively $+\epsilon$).

Definition 5.2.6. *A compact symplectic manifold with contact-type boundary is called a* compact symplectic cobordism*. The completion of a symplectic cobordism with respect to some choice of Liouville fields is called a* completed symplectic cobordism*.*

Finally, we discuss almost complex structures.

Definition 5.2.7. *Let (X, ω) be a symplectic cobordism with boundary M. Let η be a Liouville field defined in a collar neighbourhood N of M, with contact form λ and hyperplane distribution ζ. An ω-compatible almost complex structure on X is* adjusted to M for the Liouville field η *if it is of the form $\tilde{J}$ on N for a $d\lambda$-compatible J on ζ (where we are using η to identify N with a subset of the symplectisation of M). We sometimes write* η-adjusted *for brevity, or just* adjusted *with a tacit choice of η.*

An η-adjusted almost complex structure J extends cylindrically to an $\overline{\omega}$-compatible almost complex structure $\overline{J}$ on the symplectic completion $\overline{X}$. We call this the *cylindrical completion* of (X, J).

Lemma 5.2.8. *$\phi_*\overline{J}$ is an ω-compatible almost complex structure on $\mathring{X}$.*

Proof. If $v = a\partial_{\mathbb{R}} + bR + \xi$ where $\partial_{\mathbb{R}}$ is the Liouville direction, R is the Reeb field and $\xi \in \zeta$, then

$$\begin{aligned} \overline{J}(v) &= -b\partial_{\mathbb{R}} + aR + J\xi \\ \phi_*\overline{J}(v) &= -g'(g^{-1}(t))b\partial_{\mathbb{R}} + a(g^{-1})'(t)R + J\xi \end{aligned}$$

so, as $\omega = d(e^t\lambda) = e^t(dt \wedge \lambda + d\lambda)$,

$$\omega(v, \phi_*\overline{J}v) = e^t\left(a^2(g^{-1})'(t) + b^2 g'(g^{-1}(t)) + d\lambda(\xi, J\xi)\right)$$

which is positive because J is $d\lambda$-compatible and g and g^{-1} are both concave. Also,

$$\omega(\phi_*\overline{J}v, \phi_*\overline{J}w) = g'(g^{-1}(t))(g^{-1})'(t)(a_v b_w - a_w b_v) + \omega(\xi_v, \xi_w)$$

and the first term is just $(gg^{-1})'(t) = 1$, therefore

$$\omega(\phi_*\overline{J}v, \phi_*\overline{J}w) = \omega(v, w)$$

□

Neck-stretching

Given a contact-type hypersurface M in a closed symplectic manifold (X, ω), one can cut along it to obtain a compact symplectic cobordism X' with two boundary components M_1 and M_2. A Liouville field η for M restricts to two Liouville fields η_1 and η_2 on collar neighbourhoods of M_1 and M_2 respectively, so that these are boundary components of contact-type. Since η is transverse to M, one of M_1, M_2 must be convex, the other concave. Without loss of generality, suppose M_1 is convex.

Let J_1 be an ω-compatible almost complex structure on X which is adjusted to M for a Liouville field η. One forms a family J_t of ω-compatible almost complex structures on X as follows:

- Let $\mathcal{F}_s$ be the flow of η.
- Let $I_t = [-t - \epsilon, t + \epsilon]$ and define a diffeomorphism $\Phi_t : I_t \times M \to X$ by

  $$\Phi_t(s, m) = \mathcal{F}_{\beta(s)}(m)$$

  Here $\beta : I_t \to [-\epsilon, \epsilon]$ is a strictly monotonically increasing function satisfying $\beta(s) = s + t$ on $[-t - \epsilon, -t - \epsilon/2]$ and $\beta(s) = s - t$ on $[t + \epsilon/2, t + \epsilon]$.
- Equip $I_t \times M$ with the symplectic form

  $$\omega_\beta = d\left(e^{\beta(s)} \iota_\eta \omega\right) = \Phi_t^* \omega \tag{5.1}$$
- Let $\tilde{J}_t$ be the η-invariant almost complex structure on $I_t \times M$ such that $\tilde{J}_t|_\zeta = J_\zeta$.
- Glue the almost complex manifold $(X \setminus \Phi_t(I_t \times M), J)$ to $(I_t \times M, \tilde{J}_t)$ via Φ_t. By equation 5.1, the result is symplectomorphic to X, but it has a new ω-compatible almost complex structure, which we denote by J_t.

This sequence J_t of almost complex structures on X is called a *neck-stretch of* J_1 *along* M. We also consider the noncompact almost complex manifold (X_∞, J_∞) which is the cylindrical completion of (X', J_1).

Reeb dynamics

The closed orbits of the Reeb flow ϕ_t (generated by R) are important. Let $\tilde{N}_T$ denote the space of closed Reeb orbits in M with period T.

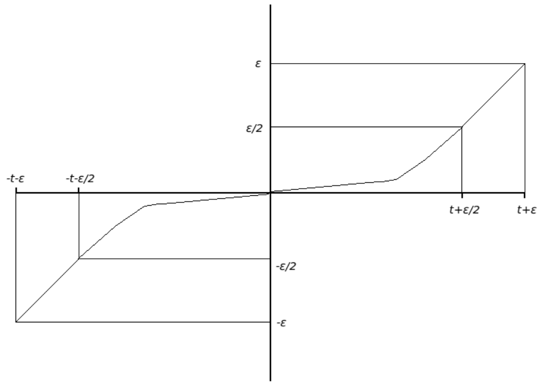


Figure 5.1: A valid function β for defining ω_β.

Definition 5.2.9. *The Reeb flow ϕ_t is of Morse-Bott type if for every T the space $\tilde{N}_T$ is a smooth closed submanifold of M with $d\lambda|_{\tilde{N}_T}$ of locally constant rank and if the linearised return map $d\phi_T - id$ is non-degenerate in the normal directions to $\tilde{N}_T$, i.e. its kernel is the tangent space to $\tilde{N}_T$.*

The minimal period of a Reeb orbit γ is the smallest period of any Reeb orbit γ' with $\gamma(\mathbb{R}) = \gamma'(\mathbb{R})$. We denote by N_T the moduli orbifold $\tilde{N}_T/S^1$ of Reeb orbits; the orbifold points correspond to Reeb orbits with period T/m covered m times.

In the Morse-Bott case, we have the following lemma ([6], lemma 3.1):

Lemma 5.2.10. *Let M be a $2n-1$-manifold with contact form λ and Reeb field R of Morse-Bott type. Let γ be a closed Reeb orbit of period T with minimal period $\tau = T/k$. Suppose $N = \tilde{N}/S^1$ is the moduli orbifold of Reeb orbits containing γ. Then there is a tubular neighbourhood V of $\gamma(\mathbb{R})$, a neighbourhood $U \subset S^1 \times \mathbb{R}^{2n-2}$ of $S^1 \times \{0\}$ and a covering map $\psi : U \to V$ such that*

- *$V \cap \tilde{N}$ is invariant under the Reeb flow,*
- *$\psi|_{S^1\times\{0\}}$ covers $\gamma(\mathbb{R})$ exactly k times,*
- *the ϕ-preimage of a periodic orbit γ' in $V \cap \tilde{N}$ is a union of circles $S^1 \times \{a\}$ where $a \in \{0\} \times \mathbb{R}^{\dim(N)} \subset \mathbb{R}^{2n-2}$,*
- *$\psi^*\lambda = f\lambda_0$, where $\lambda_0 = d\theta + \sum_{i=1}^{n-1} x_i dy_i$ is the standard contact form on $S^1 \times \mathbb{R}^{2n-2}$ and*
- *f is a positive smooth function $f : U \to \mathbb{R}$ satisfying $f|_{S^1\times\{0\}\times\mathbb{R}^{\dim(N)}} \equiv T$ and $D_{(\theta,0,a)}f = 0$ for all $(0, a) \in \{0\} \times \mathbb{R}^{\dim(N)}$.*

5.2.11 Punctured finite-energy holomorphic curves

Definitions

Let (X, ω) be a compact symplectic cobordism with boundary $M = M_+ \cup M_-$ (where M_+ denotes the union of convex components and M_- the union of concave components) and let J be an ω-compatible almost complex structure on X which is adjusted to M for a choice of Liouville fields η. Let $(\overline{X}, \overline{J})$ be the symplectic completion with its cylindrically-extended almost complex structure and denote by E_+ and E_- the set of convex (respectively concave) ends of $\overline{X}$.

Definition 5.2.12 (Energy)**.** *Let (S, j) be a closed Riemann surface and*

$$Z = \{z_1, \dots, z_k\} \subset S$$

a set of punctures. A map $F : S \setminus Z \to \overline{X}$ has energy

$$E(F) = \int_{\overline{X} \setminus E_+ \cup E_-} F^*\omega + E_\lambda(F)$$

where E_λ is given by

$$\sup_{\phi_\pm \in \mathcal{C}} \left(\int_{F^{-1}(E_+)} (\phi_+ \circ a_+)\, da_+ \wedge v_+^* \lambda_+ + \int_{F^{-1}(E_-)} (\phi_- \circ a_-)\, da_- \wedge v_-^* \lambda_- \right)$$

where $F|_{E_\pm} = (a_\pm, v_\pm)$ in coordinates on $\mathbb{R}_\pm \times M_\pm$, $\lambda_\pm$ is the contact form on $M_\pm$ and $\mathcal{C}$ consists of pairs of functions (ϕ_-, ϕ_+) where $\phi_\pm : \mathbb{R}_\pm \to \mathbb{R}$ is such that

$$\int_0^\infty \phi_+(s) ds = \int_{-\infty}^0 \phi_-(s) ds = 1$$

Definition 5.2.13 (Punctured curves)**.** *Let (S, j) be a closed Riemann surface and $Z \subset S$ a set of punctures. A punctured finite-energy $\overline{J}$-holomorphic curve is a $(j, \overline{J})$-holomorphic map $F : \dot{S} := S \setminus Z \to \overline{X}$ with finite energy. We write (S, j, Z, F) for the data of a punctured finite-energy holomorphic curve.*

We also define the oriented blow-up S^Z of S at the punctures to be the compactification of $\dot{S}$ obtained by replacing $z \in S$ with the circle Γ_z of oriented real lines in T_zS through z. This comes with a canonical circle action on Γ_z because the complex structure on T_zS allows us to define multiplication by e^{it} which lifts to Γ_z. This will be relevant later when we examine the asymptotics of punctured curves.

Asymptotics

We begin by recalling the following important observation of Hofer [21]:

Lemma 5.2.14. *Let $F = (a, v) : \mathbb{C}^* \to \mathbb{R} \times M$ be a finite-energy J-holomorphic cylinder where J is a cylindrical $d(e^t\lambda)$-compatible almost complex structure on $\mathbb{S}(M)$ for some contact form λ on M. Then there is a $T > 0$, a Reeb orbit γ and a sequence $R_k \to \infty$ such that*

$$\lim_{k\to\infty} v(R_k e^{2\pi i t}) = \gamma(Tt) \tag{5.2}$$

with convergence in $\mathcal{C}^\infty(S^1, M)$.

Suppose moreover that the Reeb flow on (M, λ) is Morse-Bott. Lemma 5.2.14 can be strengthened significantly in this setting. Suppose $F = (a, v)$ is a finite-energy cylinder and γ is a Reeb orbit for which there is a sequence R_k such that equation 5.2 holds.

Proposition 5.2.15 ([23], proposition 2.1)**.** *Let $F = (a, v)$ be a map as in lemma 5.2.14. Let Γ be the subspace of $\mathcal{C}^\infty(S^1, M)$ consisting of T-periodic Reeb orbits in the Morse-Bott family containing γ. If W is an S^1-invariant neighbourhood of Γ then there is a constant R_0 such that for all $R \geq R_0$, $v(R, \cdot) \in W$.*

Now using the coordinates around γ which we defined in lemma 5.2.10 and coordinates $(s, t) \in \mathbb{R} \times S^1 \cong \mathbb{C}^*$, the Cauchy-Riemann equations for F read:

$$\begin{aligned}
0 &= \frac{\partial z}{\partial s} + J|_\xi \frac{\partial z}{\partial t} - \frac{1}{T^2}\left(a_t - a_s J|_\xi\right) \begin{pmatrix} 0 \\ \nabla^2_{N^\perp} f \end{pmatrix} z_{N^\perp} \\
0 &= \frac{\partial a}{\partial s} - \lambda(v_t) \\
0 &= \frac{\partial a}{\partial t} + \lambda(v_s)
\end{aligned}$$

where z is the projection of v to $\mathbb{R}^{2n-2}$, $J|_\xi$ is the matrix for the cylindrical almost complex structure on the contact hyperplanes ξ and $z_{N^\perp}$ is the projection of v onto the $2n-2-\dim(N)$-dimensional subspace $\left(\{0\} \times \mathbb{R}^{\dim(N)}\right)^\perp$. Writing the first of these equations as

$$0 = \frac{\partial z}{\partial t} + A(s, t) z_{N^\perp}$$

allows us to make the following definition:

Definition 5.2.16. *The* asymptotic operator *of F is the limit*

$$A_\infty(t) = \lim_{s\to\infty} A(s, t) : L^2(S^1, M) \to L^2(S^1, M)$$

To show this is well-defined requires some further asymptotic estimates (see [23], lemma 2.2). With this in hand, the following theorem can be shown.

Theorem 10 ([23], theorems 1.2-1.3 or [6], chapter 3)**.** *Let (M, λ) be a contact manifold with Morse-Bott Reeb flow. Let J be a cylindrical adjusted almost complex structure on $\mathbb{S}(M)$. Suppose that $F = (a, v) : \mathbb{C}^* \to \mathbb{S}(M)$ is a finite-energy J-holomorphic curve and that γ is a Reeb orbit for which there is a sequence R_k such that equation 5.2 holds. Then*

$$\lim_{R\to\infty} v(Re^{2\pi it}) = \gamma(Tt)$$

as smooth maps $S^1 \to M$.

Further, if $\mathbb{R}\times\mathbb{R}^{2n-2}$ are coordinates universally covering the coordinates from lemma 5.2.10 around γ with respect to which F is represented by

$$\begin{aligned} (a, v) : [s_0, \infty) \times \mathbb{R} &\to \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{2n-2} \\ (a, v)(s, t) &= (a(s, t), \theta(s, t), z(s, t)) \end{aligned}$$

(for large s_0) then:

- *There are constants $a_0, \theta_0 \in \mathbb{R}$ and $d > 0$ such that*

$$\begin{aligned} |\partial^\beta(a(s, t) - Ts - a_0)| &\leq Ce^{-ds} \\ |\partial^\beta(\theta(s, t) - ks - \theta_0)| &\leq Ce^{-ds} \end{aligned}$$

  where β is a multi-index, C is a constant depending on the multi-index and k is the number of times the orbit of period T wraps the simple orbit of period τ with the same image. Notice that $\theta(s, t+\tau) = \theta(s, t) + k\tau$.

- *If $z \not\equiv 0$ (in which case F would be a cylinder on γ), the following asymptotic formula holds for z:*

$$z(s, t) = e^{\int_{s_0}^{s} \mu(\sigma)d\sigma} (e(t) + r(s, t)) \tag{5.3}$$

  where $r(s, t)$ tends to zero uniformly with all derivatives as s tends to infinity, $\mu : [s_0, \infty) \to \mathbb{R}$ is a smooth function which tends to a number $L < 0$ in the limit $s \to \infty$ and $e(t)$ is a vector in $\mathbb{R}^2$.

More explicitly, the number L and the vector $e(t)$ are an eigenvalue and corresponding eigenfunction for the asymptotic operator A_∞ on $L^2(S^1, \mathbb{R}^2)$.

Though we have been talking in this section about punctured cylinders in symplectisations, the theory carries through for punctured finite-energy curves in cylindrical completions of symplectic manifolds with contact-type boundary.

Proposition 5.2.17 ([7], proposition 6.2)**.** *Let X be a compact symplectic cobordism with boundary M and J an adjusted almost complex structure. Suppose the Reeb flow on M is Morse-Bott. Let $(S, j, Z, F : \dot{S} \to \overline{X})$ be a punctured finite-energy $\overline{J}$-holomorphic curve in the cylindrical completion $(\overline{X}, \overline{J})$. Then for every $z \in Z$, either F extends continuously over z to a holomorphic map or there is a neighbourhood $\mathbb{C}^* \subset \dot{S}$ of the puncture z which is asymptotic to a Reeb orbit γ_z in the sense of theorem 10 above. Consequently, embedding $\overline{X}$ as $\mathring{X} \subset X$ via the diffeomorphism ϕ from 5.2.1 above, the curve $\phi \circ F$ extends continuously to a map $S^Z \to X$, sending the circle Γ_i compactifying the puncture z to the asymptotic orbit γ_z. This map is equivariant with respect to the canonical actions of S^1 on Γ_z and (via the Reeb flow) on γ_z.*

If M consists of convex components M^+ and concave components M^-, denote by $Z^\pm$ the sets of punctures of a holomorphic curve (S, j, Z, F) which are asymptotic to Reeb orbits on $M^\pm$ and by $s^\pm$ the size of $Z^\pm$.

5.2.18 Moduli spaces

Outline

In what follows, (X, ω) is a compact symplectic cobordism with boundary M and J is an adjusted almost complex structure. Let $S \setminus Z$ be a punctured genus zero Riemann surface and $\{\rho_z\}_{z \in Z}$ be a collection of Morse-Bott manifolds of unparametrised Reeb orbits in M with representative orbits $\gamma_z \in \rho_z$. For each $x \in Z$ we pick a Reeb orbit r_z to act as a basepoint of ρ_z. A punctured finite-energy curve $F : S \setminus Z \to X$ such that the puncture z is asymptotic to an orbit γ_z from ρ_z defines a relative homology class $A \in H_2(X, \bigcup_{z \in Z} r_z)$ as soon as we choose a path in ρ_z from γ_z to r_z and consider this cylinder (sitting inside the boundary M of X) glued to the end of the curve F. In all our cases, ρ_z will be simply-connected, so the relative homology class thus defined will be independent of the choice of path. Once we have specified a collection of Morse-Bott manifolds of Reeb orbits and their basepoints, we can talk about the group $H_2(X, \bigcup_{z \in Z} r_z)$ and for a fixed class $A \in H_2(\bigcup_{z \in Z} r_z)$ we can hope to write down a moduli problem for all punctured finite-energy curves with these asymptotics defining this element of the relative homology group. The signs and multiplicities of orbits are subsumed into the notation: if a puncture is asymptotic to an orbit γ with multiplicity k then for our purposes it is asymptotic to the orbit $\gamma' = k\gamma$ and we work with the moduli space ρ' of γ'.

To such a relative homology class A one can assign a *relative first Chern class* $c_1^\Phi(A)$ once one has chosen trivialisations Φ of the contact hyperplane distributions near the orbits γ_z:

Definition 5.2.19 ([42]). *Let $S \setminus Z \to X$ be a smoothly immersed representative of $A \in H_2(X, \bigcup_{z\in Z} \gamma_z)$ and E be the pullback of the tangent bundle to $S \setminus Z$. Fix a trivialisation Φ of the contact hyperplane distributions near the orbits γ_z. There is an induced (unitary) trivialisation Φ of E over the cylindrical ends of $S \setminus Z$. If $E = L_1 \oplus \ldots \oplus L_k$ as a sum of smooth complex line bundles then define $c_1^\Phi(A)$ to be the sum $\sum_{i=1}^k c_1^\Phi(L_i)$ where $c_1^\Phi(L_i)$ is the number of zeros of a generic section of L_i which restricts to a constant non-zero section over the ends (relative to the trivialisation Φ).*

Given the data $\left(S \setminus Z, \rho_Z = \{\rho_z\}_{z\in Z}, A \in H_2(X, \bigcup_{z\in Z} r_z)\right)$ (where Z is understood as the union of positive and negative punctures $Z = Z^+ \cup Z^-$ where ρ_z consists of Reeb orbits in the convex respectively concave ends of X) we introduce the moduli space

$$\mathcal{M}^A_{s^-,s^+}(\rho_Z, J)$$

of $\overline{J}$-holomorphic maps $S \setminus Z \to \overline{X}$ representing the class A with $|Z^\pm| = s^\pm$. The complex structure on $S \setminus Z$ is allowed to vary and to give the moduli space a topology one must consider local Teichmüller slices in the space of complex structures on $S \setminus Z$ (see for example, [42]). We will ignore this technicality since it does not affect the proof of the transversality results - we can achieve transversality without perturbing the complex structure on $S \setminus Z$.

In outline, the moduli space $\mathcal{M}^A_{s^-,s^+}(\rho_Z, J)$ is the zero locus of a Cauchy-Riemann operator between suitable Banach manifolds whose linearisation is Fredholm of index

$$(n-3)\chi(S \setminus Z) + 2c_1^\Phi(A) + \sum_{z\in Z} (\pm)^z K(z)$$

where $(\pm)^z$ is the sign of the puncture z and

$$K(z) = \mu(\rho_z)(\pm)^z \frac{1}{2} \dim(\rho_z)$$

Here μ is a generalised Conley-Zehnder index for degenerate asymptotics defined in [6], chapter 5 (depending on Φ).

We also have evaluation maps

$$\mathrm{ev}_z : \mathcal{M}^A_{s^+,s^-}(\rho_Z, J) \to \rho_z$$

sending a puncture to its asymptotic Reeb orbit and

$$\mathrm{ev}_R = \prod_{z\in Z} \mathrm{ev}_z : \mathcal{M}^A_{s^+,s^-}(\rho_Z, J) \to R_Z := \prod_{z\in Z} \rho_z$$

Analytic set-up

This section follows [42]. If (S, j) is a closed Riemann surface with a finite set $Z = Z^+ \cup Z^+$ of (positive and negative) punctures then one can pick cylindrical coordinates $(s, t) : I_\pm \times S^1 \to \mathbb{A}^\pm_z$ on punctured disc neighbourhoods $\mathbb{A}^\pm_z$ of the punctures, where $I_+ = [0, \infty)$ and $I_- = (\infty, 0]$. Let S^Z be the oriented blow-up of S at Z with compactifying circles $\{\Gamma_z\}_{z\in Z}$. Given a Hermitian rank r vector bundle E over $S\setminus Z$ which extends continuously to a smooth complex vector bundle over $\bigcup_{z\in Z} \Gamma_z$, an *admissible trivialisation* of E near the ends is a smooth unitary bundle isomorphism

$$\Phi : E|_{\mathbb{A}^\pm_z} \to I_\pm \times S^1 \times \mathbb{C}^r$$

which extends continuously to a unitary trivialisation of the bundle over the compactifying circles.

Definition 5.2.20. *Let E be a bundle over $S\setminus Z$ and $\delta_Z = \{\delta_z\}_{z\in Z}$ a set of small positive numbers (these will eventually be taken to be smaller than the smallest eigenvalue of the asymptotic operator of the given puncture). A section $\sigma : S \setminus Z \to E$ is of class $W^{k,p}_{\delta_Z}$ if for each end $\mathbb{A}^\pm_z$ there is an admissible trivialisation of E over $\mathbb{A}^\pm_z$ in which the section*

$$(s, t) \mapsto e^{\pm\delta_z s} F(s, t)$$

is in $W^{k,p}(I_\pm \times S^1, \mathbb{C}^r)$.

We now define our Banach space of maps.

Definition 5.2.21. *Let S be a closed Riemann surface and $Z \subset S$ a finite set of punctures. For each $z \in Z$ assign a Morse-Bott family ρ_z of Reeb orbits of period T_z in M and a number δ_z. A map $F : \dot{S} \to \overline{X}$ is of class $W^{k,p}_{\delta_Z}$ if F is in $W^{k,p}_{loc}$ and for each $z \in Z^\pm$ there is an orbit $\gamma_z \in \rho_z$ such that in cylindrical coordinates (s, t) on an annular neighbourhood $\mathbb{A}^\pm_z$ of z in S*

$$F(s + s_0, t) = \exp_{\tilde{\gamma}(s,t)} h(s, t)$$

for s large enough, where s_0 is a constant, $\tilde{\gamma}(s,t) = (T_z s, \gamma_z(T_z t)) \in \mathbb{R}_\pm \times M$ and $h \in W^{k,p}_{\delta_z}(\mathbb{A}^\pm_z, T(\mathbb{R}_\pm \times M))$.

When $kp > 2$ the space of $W^{k,p}_{\delta_Z}$-maps $F : S \setminus Z \to \overline{X}$ is a Banach manifold $\mathcal{B}^{k,p}_{\delta_Z}$ with tangent spaces

$$T_F \mathcal{B}^{k,p}_{\delta_Z} = W^{k,p}_{\delta_Z}(F^* T\overline{X}) \oplus V_Z \oplus X_Z$$

To understand the summands $V_Z \oplus X_Z$, pick cylindrical coordinates (s,t) on each end and coordinates $\mathbb{R} \times S^1 \times \mathbb{R}^{2n-2}$ near each asymptotic orbit γ_z as in lemma 5.2.10. Let $\kappa^z_1, \kappa^z_2 \in \mathcal{C}^\infty(\mathbb{A}^\pm_z, F|^*_{\mathbb{A}^\pm_z} T\overline{X})$ be the vector fields given in local coordinates on $\mathbb{R} \times S^1 \times \mathbb{R}^{2n-2}$ by $(1,0,0)$ and $(0,1,0)$. Extend these fields by a cut-off function to the rest of $S \setminus Z$. The $2|Z|$-dimensional space they span is defined to be V_Z. The construction of X_Z is similar, but the vector fields at a given end are taken to be the constant fields in $\mathbb{R}^{2n-2}$ tangent to the Morse-Bott space parametrising Reeb orbits. X_Z can be identified with $T_{\gamma_Z} R_Z$ if γ_Z is the point in R_Z representing the Reeb orbits to which F is asymptotic.

Note that at this point, we can fix some extra data to obtain a related space. Let $Z_c \subset Z$ be a subset of punctures whose asymptotic orbit we fix to be $\gamma_z \in \rho_z$. We call these the *constrained* punctures. The corresponding Banach manifold of maps is written $\mathcal{B}^{k,p}_{\delta_Z}(Z_c)$ and the X_Z-summand in its tangent spaces is defined similarly but where one uses only the tangent vectors to Morse-Bott spaces parametrising Reeb orbits associated to unconstrained punctures.

Let η be a choice of Liouville field near M with contact form λ and Reeb vector field R. Let $\mathcal{J}^\ell$ denote the space of ω-compatible, η-adjusted, $\mathcal{C}^\ell$-differentiable almost complex structures on X. $T_J \mathcal{J}^\ell$ is the Banach space of $\mathcal{C}^\ell$-sections Y of the endomorphism bundle $\mathrm{End}(T\overline{X})$ such that $Y\overline{J} + \overline{J}Y = 0$, $\overline{\omega}(Yv, w) + \overline{\omega}(v, Yw) = 0$, $Y(\xi) \subset \xi$ (where ξ is the contact distribution) and $Y(R) = Y(\eta) = 0$. This space is written $\mathcal{C}^\ell(\mathrm{End}(T\overline{X}, J, \eta, \omega))$.

Over the product $\mathcal{J}^\ell \times \mathcal{B}^{k,p}_{\delta_Z}$ define the Banach bundle $\mathcal{E}$ whose fibre at (J,F) is the space

$$W^{k-1,p}_{\delta_Z}(\Lambda^{0,1}_j \otimes_{(j,\overline{J})} F^* T\overline{X})$$

and let σ be the section

$$\sigma(J,F) = dF + \overline{J} \circ dF \circ j$$

A zero (J,F) of σ is precisely a finite-energy punctured $\overline{J}$-holomorphic curve.

Definition 5.2.22. *We define the universal moduli space of simple finite-energy curves to be the space $\mathcal{M}^*(A, S, Z, \rho_Z, \mathcal{J}^\ell)$ of $(J,F) \in \sigma^{-1}(0)$ such that F does not factor through a multiple cover.*

We have the following basic transversality results. The proofs are closely modelled on [28], propositions 3.2.1 and 3.4.2. and can be found in appendix A.2 below.

Proposition 5.2.23. *Let e_z denote the smallest eigenvalue of the asymptotic operator A_z assigned to the puncture z. For $\delta_Z \in \prod_{z\in Z}(0, e_z)$ the universal moduli space is a separable $\mathcal{C}^{\ell-k}$ Banach submanifold of $\mathcal{J}^\ell \times \mathcal{B}^{k,p}_{\delta_Z}$ and the projection map to $\mathcal{J}^\ell$ is a $\mathcal{C}^{\ell-k}$-smooth Fredholm map whose index is given in theorem 11 below.*

The condition on δ_Z is necessary for the Fredholm theory to work and implies the stated index formula. Only the transversality aspects of this proposition are proved below; the underlying Fredholm theory is referenced to the work of Schwarz [32] (see also [41]).

Proposition 5.2.24. *Every point of $R_Z := \prod_{z\in Z}\rho_z$ is a regular value of the evaluation map*

$$ev_R : \mathcal{M}^*(A, S, Z, \rho_Z, \mathcal{J}^\ell) \to R_Z$$

Let $\pi_{\mathcal{J}}$ denote the projection of $\mathcal{M}^*(A, S, Z, \rho_Z, \mathcal{J}^\infty)$ to $\mathcal{J}^\infty$ where $\mathcal{J}^\infty$ denotes the subset of smooth almost complex structures. Note that by elliptic regularity the corresponding universal moduli space consists of smooth curves. The above transversality results together with the Sard-Smale theorem will prove the following results (see section A.2).

Theorem 11. *For J in a Baire set $\mathcal{J}_{reg} \subset \mathcal{J}^\infty$ the space*

$$\mathcal{M}^A_{s^+,s^-}(\rho_Z, J) = \pi^{-1}_{\mathcal{J}}(J)$$

is a smooth manifold of dimension

$$(n-3)\chi(S \setminus Z) + 2c_1^\Phi(A) + \sum_{z\in Z}(\pm)^z K(z)$$

where $(\pm)^z$ is the sign of the puncture z and

$$K(z) = \mu(\rho_z)(\pm)^z \frac{1}{2}\dim(\rho_z)$$

Here μ is a generalised Conley-Zehnder index for degenerate asymptotics defined in [6], chapter 5 (depending on Φ).

Theorem 12. *Suppose all asymptotic orbits are simple and fix a submanifold $P \subset R_Z$. For J in a Baire set $\mathcal{J}^P_{reg} \subset \mathcal{J}_{reg}$ the evaluation map*

$$ev_R : \mathcal{M}^A_{s^+,s^-}(\rho_Z, J) \to R_Z$$

is transverse to P.

5.2.25 Compactness

Let (X_∞, J_∞) be the non-compact almost complex manifold obtained by stretching the neck around some contact-type hypersurface $M \subset X$ and let X^k_∞ be the union

$$X_\infty \cup \coprod_{i=1}^{k} \mathbb{S}_i(M)$$

$\mathbb{S}_i(M)$ is understood to be equipped with the same cylindrical almost complex structure as the ends of X_∞. If M is a separating hypersurface cutting X into two compact pieces then we denote these by W and V, where W has convex boundary and V has concave boundary.

Consider a collection of finite-energy punctured J_∞-holomorphic curves $F_\nu : \Sigma_\nu \setminus Z_\nu \to \mathbb{S}_\nu(M)$ (for $\nu \in \{1, \ldots, k\}$) and $F_0 : \Sigma_0 \setminus Z_0 \to X_\infty$. We allow these to have sets of marked points K_ν and sets of special marked pairs $D_\nu = \{\overline{d}_i, \underline{d}_i\}$ for which $F_\nu(\overline{d}_i) = F_\nu(\underline{d}_i)$ (creating a node d^ν_i). Let $\Sigma^{Z_\nu}_\nu$ denote the oriented blow-up of Σ_ν at Z_ν with compactifying circles Γ^ν_z, $z \in Z_\nu$. Let $\Gamma^\nu_\pm$ denote the union of the compactifying circles at positive/negative punctures in the ν-th curve.

Recall from section 5.2.1 that X_∞ was diffeomorphic to $X \setminus M$ via a diffeomorphism ϕ. Let $N \cong [-\epsilon, \epsilon] \times M$ be a closed Liouville collar of M and use the Liouville flow to define a new diffeomorphism ϕ^0 from X_∞ to $X \setminus N$. Define $N_\nu \subset N$ for $\nu = 1, \ldots, k$ to be the subset $[-\epsilon + \frac{2(\nu-1)\epsilon}{k}, \epsilon, +\frac{2\nu\epsilon}{k}] \times M$ and notice that the symplectic completion of this is the symplectisation of M. Therefore there is a diffeomorphism ϕ^ν identifying $\mathbb{S}_\nu M$ with the interior of N_ν. Therefore the space X^k_∞ maps into X via the union of these diffeomorphisms with image the complement of a collection of embedded copies of M. See figure 5.2.

Definition 5.2.26. *The data $(F_\nu, \Sigma_\nu, Z_\nu, K_\nu, D_\nu)$ defines a level k-holomorphic building in X^k_∞ if there are a sequence $\{\Phi^\nu : \Gamma^\nu_+ \to \Gamma^{\nu+1}_-\}^k_{\nu=1}$ of orientation-reversing diffeomorphisms (orthogonal on each boundary component) for which the compactifications of the maps $\phi^\nu \circ F_\nu$ glue to give a piecewise smooth map*

$$\dot{F} : \Sigma^Z := \bigcup_{\Phi^\nu} \Sigma^{Z_\nu}_\nu \to X$$

The genus *of a holomorphic building is the genus of the topological surface Σ^Z.*

In the case where M is separating, we write F^W and F^V for the W- and V-parts respectively, so $\Sigma_0 = \Sigma^V \cup \Sigma^W$ and $F^W = F_0|_{\Sigma^W}$, $F^V = F_0|_{\Sigma^V}$.

Definition 5.2.27. *A marked level-k holomorphic building is* stable *if every constant*

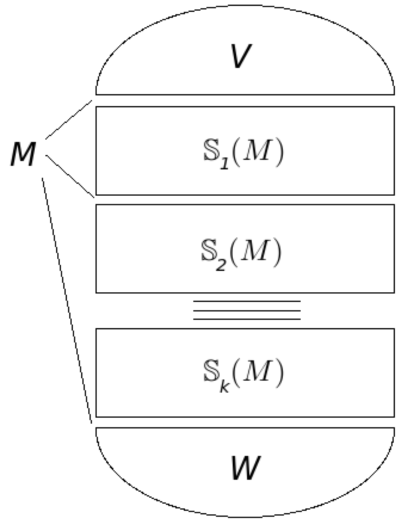


Figure 5.2: The decomposition of X into W-, V- and symplectisation parts.

component has at least three marked points and if there is no level ν for which all components of F_ν are unmarked Reeb cylinders.

For the relevant notions of equivalence and convergence for holomorphic buildings, we refer to the papers [7] and [10] as the definitions are very involved. Instead, we remark that one can topologise the space of equivalence classes of stable level-k genus g buildings with μ marked points and $s^{\pm}$ positive/negative punctures with the topology of *Gromov-Hofer convergence.* For us the most important property in the definition of Gromov-Hofer convergence is the following:

Lemma 5.2.28. *If $F_{\nu,j} : \Sigma_\nu \setminus Z_\nu \to X^k_\infty$ is a sequence of holomorphic buildings which Gromov-Hofer converge to a level-k' building $F_\nu : \Sigma_\nu \setminus Z_\nu \to X^{k'}_\infty$ then the compactified curves $\dot{F}_j : \Sigma_j^{Z_j} := \bigcup_{\Phi_j^\nu} \Sigma_{j,\nu}^{Z_{j,\nu}} \to X$ converge in $\mathcal{C}^\infty_{loc}$ on compact subsets of the image of X^k_∞ in X and in $\mathcal{C}^0$ everywhere.*

The reason for introducing these notions is the following set of compactness results, rephrased from the all-purpose theorem of [7]:

Theorem 13 ([7], theorem 10.3)**.** *Let (X,ω) be a closed symplectic manifold, M a contact-type hypersurface and J_1 an ω-compatible almost complex structure adjusted to some choice of Liouville fields near M. Let J_t denote the family of almost complex structures obtained by neck-stretching J_1 along M and let $u_j : S_j \to X$ be a sequence of J_{t_j}-holomorphic curves with ω-energy bounded from above. Then there exists a subsequence u_{j_ℓ}, a number k and a level-k holomorphic building F in X^k_∞ such that u_{j_ℓ} Gromov-Hofer converges to F.*

Theorem 14 ([7], theorem 10.2)**.** *Let (X, ω) be a compact symplectic cobordism and J an ω-compatible almost complex structure adjusted to some choice of Liouville fields along the boundary. Then a sequence of $\overline{J}$-holomorphic buildings of level-k with bounded energy in $\overline{X}$ has a Gromov-Hofer convergent subsequence whose limit is a level k'-holomorphic building for some k'.*

5.3 Neck-stretching for Lagrangian spheres in Del Pezzo surfaces

In this section, (X, ω) will be a symplectic Del Pezzo surface $\mathbb{D}_n$ and $L \subset X$ will be a Lagrangian sphere. To apply the machinery developed in the previous section, we extract from this data:

- A contact-type hypersurface M enclosing a neighbourhood W of L,
- An almost complex structure J_1 on X which is very explicitly given on W,
- A family of almost complex structures J_t arising from neck-stretching of J_1 along M.

We also need to understand specific properties of holomorphic curves in the symplectic completion $\overline{W}$.

5.3.1 The neck-stretching data

As L is a Lagrangian 2-sphere, Weinstein's neighbourhood theorem guarantees the existence of a neighbourhood $\widetilde{W}$ of L which is symplectomorphic to a neighbourhood of the zero-section in T^*S^2 with its canonical symplectic structure.

More explicitly, write T^*S^2 in coordinates:

$$T^*S^2 = \{(u, v) \in \mathbb{R}^3 \times \mathbb{R}^3 : |u| = 1, u \cdot v = 0\}$$

with canonical symplectic form $\omega_{\text{can}} = d\lambda_{\text{can}}$ where

$$\lambda_{\text{can}} = \sum_{j=1}^{3} v_j du_j$$

and let H be the Hamiltonian function

$$H(u, v) = \frac{1}{2}|v|^2$$

The Hamiltonian flow generated by H is the cogeodesic flow on the round sphere and for $c > 0$ the c-level set of H is a contact-type hypersurface. The contact plane distribution is

$$\zeta_{(u,v)} = \langle (u \times v, 0), (0, u \times v) \rangle$$

and the Reeb vector field is $R = (v, 0)$, the ω-dual to dH. Write M_c for $H^{-1}(c)$ and W_c for $H^{-1}([0, c])$.

Suppose $\Psi : \widetilde{W} \to W_{2r}$ is the symplectomorphism given by Weinstein's neighbourhood theorem, taking L to the zero-section. The hypersurface $M = \Psi^{-1}(M_r)$ is a contact-type hypersurface in X, bounding $W = \Psi^{-1}(W_r)$. The hypersurface M is diffeomorphic to $\mathbb{RP}^3$.

We now write down an explicit ω_{can}-compatible complex structure I on T^*S^2 which we can restrict to W_{2r}, pull-back via Ψ to $\widetilde{W}$ and extend arbitrarily but ω-compatibly to the rest of X. The complex structure is obtained by identifying T^*S^2 with the affine quadric

$$\mathcal{Q} = \{x_1^2 + x_2^2 + x_3^2 = 1\} \subset \mathbb{C}^3$$

via the map $\varpi : T^*S^2 \to \mathcal{Q}$ given in coordinates by

$$x_j = u_j \cosh(|v|) + iv_j \sinh(|v|)/|v|$$

under which the Liouville form on $\mathcal{Q}$ inherited from $\mathbb{C}^3$ pulls back to (a positive multiple of) the canonical Liouville 1-form on T^*S^2:

$$\varpi^* \lambda_{\mathcal{Q}} = \frac{\cosh(|v|) \sinh(|v|)}{|v|} \lambda_{\text{can}}$$

Hence the pull-back of the complex structure on the quadric preserves the contact distribution on M and sends the canonical Liouville field $\eta = (0, v)$ to a rescaled Reeb field $(fv, 0)$. Thus I is ω-positive and ϖ identifies the Lagrangian zero-section with the real part of the quadric. We will interchangeably refer to these both as L.

Definition 5.3.2. *Given a Weinstein neighbourhood* Ψ*, a* neck-stretching datum *for* L *consists an extension of* $\Psi^* I$ *to an* ω*-compatible almost complex structure* J_1 *on* X*. Note that such structures are* η*-adjusted on the cylindrical neck* $\Psi^{-1}(W_{2r} \setminus W_r)$*. Let* $\mathcal{J}_1$ *denote the space of neck-stretching data.*

Apply the neck-stretching procedure from section 5.2.1 to $J_1 \in \mathcal{J}_1$ along the hypersurface M_r. The result is a sequence $\{J_t\}_{t \in [1,\infty)}$ of ω-compatible almost complex structures and we denote by (X_∞, J_∞) the noncompact almost complex, symplectic manifold with cylindrical ends:

$$\overline{W} \cup \overline{V}$$

where V is the closure of the complement of W in X. We also write X_∞^k for the manifold

$$X_\infty \cup \coprod_{i=1}^{k} \mathbb{S}_i(M)$$

where $\mathbb{S}_i(M)$ is the symplectisation of M (labelled by the integer i).

We will be interested in understanding the limits of J_t-holomorphic curves from the families constructed in section 4.3 as $t \to \infty$ for a neck-stretch J_t. To that end, let us first examine punctured finite-energy holomorphic curves in the $\overline{W}$-part of X_∞, namely the affine quadric surface.

5.3.3 Compactifying punctured curves

Asymptotics for the affine quadric $\mathcal{Q}$

In the case of the affine quadric, the Reeb flow is actually a Hamiltonian circle action (see section 5.3.1) so the manifold of periodic orbits of period T fills the whole of M. The local model in theorem 10 near a periodic orbit is just $S^1 \times \mathbb{R}^2$ with the standard contact form $\lambda = d\theta + xdy$. The Reeb orbits are then circles of constant (x, y) and the linearised return map is the identity. This implies that the relevant self-adjoint operator A in this case is just $-J_0\frac{d}{dt}$, whose eigenvalues L are integer multiples of 2π and whose eigenfunctions are $f(t) = (\cos(Lt), \sin(Lt))$.

Compactifying

Recall that the symplectic completion $\overline{W}$ can be embedded in the original compact symplectic manifold (W, ω) with boundary M by a diffeomorphism ϕ (see section 5.2.1). Furthermore, $\phi_*\overline{J}$ is compatible with ω. In the case of the affine quadric, one may take W to be the closed r-sublevel set of the Hamiltonian H. Having a Hamiltonian circle action on this level set means we may perform a symplectic cut to obtain a closed symplectic manifold:

$$\wp(W) := (W \setminus M) \cup \left(M//S^1\right)$$

(see Lerman [26]). In fact, $\wp(W)$ is a monotone $S^2 \times S^2$, and the compactification locus $M//S^1$ corresponds to the diagonal sphere $\Delta = \{(x, x) : x \in S^2\}$. The symplectic form on the symplectic cut is proportional to the Poincaré-dual of Δ.

Given a finite-energy punctured holomorphic curve $F : S \setminus Z \to \overline{W}$, the map $\phi \circ F$

extends continuously to a map $\dot{F} : S^Z \to W$. On M, the image of F' is a collection of closed Reeb orbits and therefore F' descends to a continuous map $\dot{F} : S \to \wp(W)$. Since F is holomorphic, ω integrates positively over $\dot{F}(S \setminus Z)$ and the singular cycle represented by $\dot{F}$ satisfies:

$$[\Delta] \cdot [\dot{F}] > 0$$

In terms of the basis $[S^2] \times \{0\}, \{0\} \times [S^2]$ for $H_2(\wp(W), \mathbb{Z})$, writing $[\dot{F}] = (a, b)$, this implies $a + b > 0$.

Remark 5.3.4. *Let F_1 and F_2 be two finite-energy punctured holomorphic curves in $\overline{W}$ with Γ_1 and Γ_2 their sets of asymptotic Reeb orbits. Suppose that if $\gamma \in \Gamma_1$ then there is no γ' in Γ_2 with the same image. Then the compactifications $\dot{F}_1$ and $\dot{F}_2$ only intersect in $\phi(\overline{W})$ and their intersections are precisely the images under ϕ of their intersections in $\overline{W}$.*

5.3.5 Relative first Chern class

An equivalent point of view of the symplectic cut of the affine quadric $\mathcal{Q}$ is the projective quadric surface

$$\{z_0^2 + z_1^2 + z_2^2 + z_3^2 = 0\} \subset \mathbb{CP}^3$$

If $\Delta = \{z_0 = 0\}$ denotes a hyperplane section in Q then the multiple divisor 2Δ is in the anticanonical linear system. Hence the determinant line bundle $\det(TQ)$ has a holomorphic section σ that is non-vanishing on $\mathcal{Q}$.

In general, if $U \subset X$ is an open subset of a symplectic manifold which is symplectomorphic to a subset of $\mathcal{Q}$ and J is an ω-compatible almost complex structure on X agreeing with the restriction of the affine quadric complex structure on U then $\det(TX)$ can be J-unitarily trivialised on the open set U using this section σ. Thus with respect to this trivialisation Φ, the relative first Chern class $c_1^{\Phi}(u)$ of a punctured holomorphic curve u in $\mathcal{Q}$ is 0.

One can define the first Chern class of a curve in the complement of the subset U using this trivialisation near the boundary and trying to extend constant sections over the interior. If f is a curve in X which has a component u in U and v in the complement of U then

$$c_1^{\Phi}(u) + c_1^{\Phi}(v) = c_1(f)$$

In particular, suppose that J_t is a sequence of ω-compatible almost complex structures on X arising from a neck-stretch and f_t a Gromov-Hofer convergent sequence of J_t-holomorphic curves in a fixed homology class C. Let f_∞ denote the J_∞-holomorphic limit building with levels: u landing in the completion of U, s_k landing in the symplectisation at level k and v landing in the completion of the complement of U. Then, since the determinant line bundle can be unitarily trivialised on U and on the symplectisation levels,

$$c_1^{\Phi}(v) = c_1(C).$$

5.3.6 Punctured curves in the affine quadric: examples and properties

The easiest way to obtain examples of punctured curves in T^*S^2 is to view it as the affine quadric $\mathcal{Q} \subset \mathbb{C}^3$ as in section 5.3.1. It is well-known that every point $p \in \mathcal{Q}$ lies on exactly two complex lines α_p and β_p in $\mathbb{C}^3$ with $\alpha_p, \beta_p \subset \mathcal{Q}$. Globally the fibration over $\mathcal{Q}$ whose fibre at p is the two point set $\{\alpha_p, \beta_p\}$ is a 2-to-1 cover of $\mathcal{Q}$. Since $\pi_1(\mathcal{Q}) = 0$, the total space of this fibration has two components, corresponding to two distinct families of planes. Another way to see these families of planes is as follows: Given a choice of orientation on $L = \mathrm{Re}\ \mathcal{Q}$ we can define α-planes to be planes which intersect L positively and β-planes to be planes which intersect L negatively.

Lemma 5.3.7. *Thought of as punctured holomorphic planes, each α- (respectively β-) plane is asymptotic to a single simple Reeb orbit on M. There is a unique α- (respectively β-) plane asymptotic to each Reeb orbit.*

Proof. For the affine quadric, the projective compactification

$$Q = \{z_0^2 + z_1^2 + z_2^2 + z_3^2 = 0\}$$

agrees with the symplectic cut, the projective compactification locus being the hyperplane section $\{z_0 = 0\}$. Affine lines projectively compactify to projective lines and therefore intersect the compactification locus in exactly one point transversely. This point is the representative of the unique asymptotic Reeb orbit, and every point on $\{z_0 = 0\}$ lies on a projective line of $\mathbb{CP}^3$ contained in Q. □

Remark 5.3.8. *In fact, if F is a finite-energy punctured holomorphic curve asymptotic to a Reeb orbit γ of period $k\tau$ and minimal period τ then we can find the local intersection number of $\dot{F}$ and Δ by looking at the model case of an α-plane asymptotic to γ, covered k-times by $\mathbb{C}$ with branching over the origin. This projectively compactifies to a k-fold branched cover of a projective line, ramified at 0 and ∞, and has local intersection number k with the compactification locus Δ.*

Lemma 5.3.9. *Any finite-energy holomorphic plane asymptotic to a single simple Reeb orbit γ must be an α- or β-plane.*

Proof. If π is neither an α- nor a β-plane then, since these planes foliate $\mathcal{Q}$, π must intersect an α-plane A and a β-plane B neither of which are asymptotic to γ. Furthermore, by positivity of intersections (which is a local property of holomorphic curves and therefore holds even in the current noncompact setting) this intersection contributes positively to the intersection of π with A and B. By remark 5.3.4 the compactifications satisfy

$$\dot{\pi} \cdot A > 0 \ , \ \dot{\pi} \cdot B > 0$$

In terms of the basis $[S^2] \times \{0\}, \{0\} \times [S^2]$ for $H_2(\wp(W), \mathbb{Z})$, writing $[\dot{\pi}] = (a, b)$, this implies $a + b \geq 2$. But π is a plane asymptotic to a single simple Reeb orbit so the intersection number of $\dot{\pi}$ and Δ is 1 by remark 5.3.8. □

Lemma 5.3.10. *Let $F : S \setminus Z \to \mathcal{Q}$ be a punctured finite-energy holomorphic curve with a set Γ of asymptotic Reeb orbits. If A is an α-plane and B a β-plane, each asymptotic to some $\gamma' \notin \Gamma$ then either F covers an α- or β- plane or else it intersects both A and B.*

Proof. Suppose F does not cover an α- or β- plane and (without loss of generality)

$$F(S \setminus Z) \cap A = \emptyset$$

Then $[\dot{F}] \cdot [\dot{A}] = 0$. However the α-planes foliate $\mathcal{Q}$ so the image of F must intersect some α-plane A' whose asymptotic orbit is not contained in Γ. This intersection would be positive which would contradict $[\dot{F}] \cdot [\dot{A}] = 0$. □

5.3.11 Index formulas

Richard Hind [19], lemma 7, calculates the Conley-Zehnder indices of the Reeb orbits for this contact form on M to be $2\mathrm{cov}(\gamma)$ where $\mathrm{cov}(\gamma)$ denotes the number of times the orbit γ in question wraps around a simple Reeb orbit. We also know that the relative first Chern class of such a curve vanishes (see section 5.3.5). Therefore for genus 0, s^+-punctured curves in W, the index formula for the expected dimension of their moduli spaces reads:

$$\mathbb{E}\dim = 2(s^+ - 1) + \sum_{i=1}^{s^+} 2\mathrm{cov}(\gamma_i)$$

and for genus 0, s^--punctured curves F in V, the index formula becomes

$$\mathbb{E}\dim = 2(s^- - 1 + c_1^{\Phi}(F^*TV)) - \sum_{i=1}^{s^-} 2\mathrm{cov}(\gamma_i)$$

5.4 Analysis of limit-buildings: binary case

Let L be a binary Lagrangian sphere in $\mathbb{D}_n$, $n = 2, 3, 4$, in the homology class $E_1 - E_2$ (for notational simplicity). Let Ψ a choice of Weinstein neighbourhood and $\mathcal{J}_1$ the space of neck-stretching data for L. In this section we will analyse the J_∞-holomorphic buildings obtained as limits of sequences of stable J_t-curves from the families constructed in section 4.3.

Let $E_k(J)$, $S_{k\ell}(J)$ denote the J-holomorphic exceptional spheres in the homology classes E_k and $S_{k\ell}$. We will prove the following proposition:

Proposition 5.4.1. *For generic neck-stretching data $J_1 \in \mathcal{J}_1$ and large t,*

- $E_k(J_t) \cap L = \emptyset$ *if* $k \neq 1, 2$,
- $S_{ij}(J_t) \cap L = \emptyset$ *unless* $|\{i, j\} \cap \{1, 2\}| = 1$,
- *If $i = 3, 4$ there is a smooth J_t-holomorphic curve homologous to $H - E_i$ disjoint from L,*
- *There is a smooth J_t-holomorphic curve homologous to H disjoint from L.*

Our limit analysis will actually prove this for $t = \infty$, but the nature of Gromov-Hofer convergence then implies the proposition.

5.4.2 Exceptional spheres

We postpone analysis of the exceptional classes E_3, E_4 and S_{34} to the end of the section and first examine the limit of a convergent subsequence $E(J_{t_j})$ when E is an exceptional class E_1, E_2 or S_{12}. Such a sequence exists by the SFT compactness theorem 13. Let E_∞ denote the limit building in X^k_∞ for some k. Recall that its V- and W-parts are denoted E^V_∞ and E^W_∞ respectively.

Lemma 5.4.3. E^V_∞ *is non-empty, simple and connected.*

Proof. E^V_∞ is non-empty because the maximum principle for holomorphic curves forbids closed holomorphic curves in W. The discussion in section 5.3.5 implies that the relative first Chern class

$$c_1^\Phi(E^V_\infty) = c_1(E) = 1.$$

Any finite-energy punctured curve v in V has $c_1^\Phi(v) > 0$ since one could glue it continuously to a collection of finite-energy planes in W to obtain a singular cycle C in X which is symplectic away from M so that $\omega(C) > 0$. Since $[c_1(X)] = [\omega]$ and $c_1^\Phi(u) = 0$ for a holomorphic curve u in W, $c_1^\Phi(v) = c_1(C) = \omega(C) > 0$. Therefore each component of E_∞^V contributes positively to $c_1^\Phi(E_\infty^V) = 1$, but 1 is primitive amongst relative Chern classes. Therefore E_∞^V is connected. If v' is a branched k-fold cover of v then $c_1^\Phi(v') = kc_1^\Phi(v)$, so E_∞^V is simple. $\square$

Remark 5.4.4. *By theorem 11, we can choose a neck-stretching datum J_1 such that $J_\infty|_V$ is regular for E_∞^V. In fact, since there is a countable number of possible connected topologies for the domain of E_∞^V and a finite number of possible relative homology classes for each of those giving $c_1^\Phi = 1$, we can take an intersection of the Baire sets for each Fredholm problem and what remains is a Baire set in $\mathcal{J}_1$ making all genus 0 connected $J_\infty|_V$-holomorphic punctured curves in V with $c_1^\Phi = 1$ regular.*

In this generic case, the dimension formula for the moduli space of punctured finite-energy curves in V containing E_∞^V becomes:

$$\begin{aligned}
\dim(\mathcal{M}(S, J_\infty, E_\infty^V)) &= -2 + 2s^- + 2c_1^\Phi(E_\infty^V) - 2\sum_{i=1}^{s^-} \mathrm{cov}(\gamma_i) \\
&= 2\left(s^- - \sum_{i=1}^{s^-} \mathrm{cov}(\gamma_i)\right) \\
&\leq 0
\end{aligned}$$

with equality if and only if $\mathrm{cov}(\gamma_i) = 1$ for all γ_i, where $\{\gamma_i\}_{i=1}^{s^-}$ is the set of Reeb orbits to which E_∞^V is asymptotic at the punctures $\{z_i\}_{i=1}^{s^-}$. Thus, generically, the asymptotic Reeb orbits are simple.

Remark 5.4.5. *This means that for generic neck-stretching data, all moduli spaces of $c_1^\Phi = 1$, genus 0 punctured finite-energy curves in V are zero-dimensional and all such curves have simple Reeb asymptotics. By theorem 12, one can also ensure that the puncture-evaluation maps from such moduli spaces to products of ρ are transverse to all strata of the multi-diagonal. That is, if a $c_1^\Phi = 1$, genus 0 curve has k punctures, the map sending its moduli space to ρ^k is transverse to all smooth strata of the subset $\{(\gamma_1, \ldots, \gamma_k) : \gamma_i = \gamma_j$ for some $i, j\}$. In particular, for generic neck-stretching data, a $c_1^\Phi = 1$, genus 0 punctured finite-energy curve in V has* distinct*, simple Reeb asymptotics.*

Lemma 5.4.6. *Any part of the limit E_∞ which lands in a symplectisation component of X_∞^k is a cylinder on its asymptotic Reeb orbit.*

Proof. The λ-energy of a finite-energy curve $F = (a, v) : S \setminus Z \to \mathbb{R} \times M$ in the symplectisation is non-negative, which entails

$$\int v^* d\lambda \geq 0$$

By Stokes's theorem this integral is the sum of the periods of the asymptotic Reeb orbits, weighted ± 1 according to whether F is asymptotic to $-\infty \times \gamma$ or $+\infty \times \gamma$. Let $s^{\pm}$ denote the number of positive (respectively negative) Reeb orbits to which F is asymptotic.

Write E_∞^ℓ for the part of E_∞ landing in $\mathbb{S}_\ell(M)$ and let $\ell = m$ be the top level of the symplectisation. We know by what was said above that the positive asymptotics of E_∞^m are simple Reeb orbits. Therefore for $F = E_\infty^\ell$, the inequality above becomes

$$s^+ - \sum_{i=1}^{s^-} \mathrm{cov}(\gamma_i^-) \geq 0$$

and in particular, $s^+ \geq s^-$.

Now E_∞^V has a single component, so if $s^+ > 1$ then E_∞^m will connect two of the negative asymptotic orbits of E_∞^V, but the genus of E_∞ is zero. Hence $s^+ = 1$ and so $s^- \leq 1$. Since $M \cong \mathbb{RP}^3$ and a single Reeb orbit represents a nontrivial element of $\pi_1(\mathbb{RP}^3) \cong \mathbb{Z}/2$, $s^- \neq 0$ or else E_∞^m would be a nullhomotopy of the Reeb orbit. Hence $s^- = 1$ and E_∞^m is a cylinder with zero energy. This implies that it is a Reeb cylinder as claimed.

Inductively applying this argument to the lower symplectisation levels allows us to deduce the lemma for all ℓ. □

Finally, we consider E_∞^W. Topologically, it must consist of finite-energy planes in order for the building E_∞ to have genus zero. Since E_∞^V has simple negative asymptotics and the intermediate levels E_∞^ℓ are cylindrical, the positive asymptotics of E_∞^W are also simple. In lemma 5.3.9, we classified finite-energy planes with simple asymptotics in W. They were either α- or β-planes. In summary:

Proposition 5.4.7. *If $E(J_{t_j})$ is a convergent sequence of J_{t_j}-holomorphic exceptional spheres for a neck-stretch J_t then the limit building consists of:*

- *E_∞^V, a connected punctured finite-energy J_∞-holomorphic sphere in V asymptotic to a finite collection of simple Reeb orbits $\{\gamma_i\}$ on M,*
- *E_∞^W, a collection of α- or β-planes in W asymptotic to the Reeb orbits $\{\gamma_i\}$.*

The components of the buildings in symplectisation levels would necessarily be cylindrical, but these are ruled out by the definition of stability for a holomorphic building since there are no marked points.

Notice that since E_1 and E_2 have homological intersection zero, their J_∞-holomorphic representatives have no isolated intersections. Otherwise, for large t, there would be positively intersecting $E_1(J_t)$ and $E_2(J_t)$. Since the W-parts of their limits must be non-empty (as each class E_i has non-trivial homological intersection with $L \subset W$), and consist of α- and β- planes, we deduce:

Lemma 5.4.8. *The punctured curves $E^W_{1,\infty}$ and $E^W_{2,\infty}$ consist of α- and β-planes all of which are asymptotic to* the same Reeb orbit R.

Proof. Let A be an α-plane which is a component of $E^W_{1,\infty}$ and B a β-plane which is a component of $E^W_{2,\infty}$. Such components must exist since $E_1 \cdot L = -1$ and $E_2 \cdot L = 1$. If A and B were not asymptotic to the same Reeb orbit then they would have an isolated point of intersection which is disallowed as remarked above. Therefore they have a common asymptotic orbit R. Similarly, any β-plane in $E^W_{1,\infty}$ (respectively α-plane in $E^W_{2,\infty}$) which was not asymptotic to R would show up as a self-intersection of $E_1(J_t)$ (respectively $E_2(J_t)$) for large t. But by the adjunction formula, we saw these closed curves were embedded. $\square$

For the other exceptional classes E_3, E_4 and S_{34}, the same analysis carries through except that the W-part may be empty. Again by positivity of intersections, any α- or β-plane in E^W_3, E^W_4 or S^W_{34} must be asymptotic to R.

Proposition 5.4.9. *$E^W_{1,\infty}$ and $E^W_{2,\infty}$ consist of a single β- and α-plane respectively. $E_{3,\infty}$, $E_{4,\infty}$ and $S_{34,\infty}$ have no W-component.*

Proof. Let E stand for any of these classes. We have noted that E^W_∞ (if non-empty) consists of α- or β-planes asymptotic to a given Reeb orbit R. Therefore E^V_∞ has all its asymptotic orbits equal to R. Suppose E^V_∞ has k punctures. Suppose $k > 1$. By remark 5.4.5, the evaluation map from the moduli space of E^V_∞ to the product ρ^k is transverse to the various strata of the multi-diagonal for generic neck-stretching data J_1. Since ρ has dimension 2, the multi-diagonal has top strata of codimension 2. But the moduli space is zero-dimensional and ρ^k has dimension $2k > 2$ so this transversality means that the image of evaluation map is disjoint from the multi-diagonal. This contradicts the fact that all asymptotic orbits are equal to R. Therefore $k = 0$ or 1. If $k = 1$ then E must have intersection number ± 1 with L (as E^W_∞ consists of a single α- or β-plane). If $k = 0$ then E must have zero intersection with L. Since the classes E_3, E_4, S_{12} and S_{34} have zero intersection with L they fall into this category, while the classes E_1 and E_2 must have $k = 1$. $\square$

The final point is to show that $S_{12,\infty}$ has no W-part.

Lemma 5.4.10. $S^W_{12,\infty} = \emptyset$.

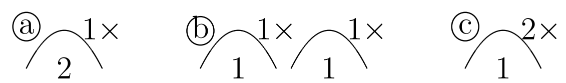
Proof. $S^W_{12,\infty}$ consists of a union of α- and β-planes. Each of these contributes ± 1 respectively to the intersection number $S_{12} \cdot [L]$. Since this intersection number vanishes, if $S^W_{12,\infty} \neq \emptyset$ then it must contain at least one α-plane and one β-plane. If these were asymptotic to different Reeb orbits then they would intersect and for large T, $S_{12}(J_T)$ would not be embedded, contradicting lemma 4.3.6. However, a transversality argument like the one in the preceding lemma will rule out the possibility that $S^V_{12,\infty}$ has two punctures asymptotic to the same Reeb orbit. Therefore the W-part must be empty. $\square$

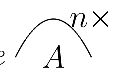
5.4.11 The classes $H - E_3$, $H - E_4$

The purpose of this section is to prove that (for generic neck-stretching data) if $i = 3, 4$ there is a smooth J_t-holomorphic curve homologous to $H - E_i$ disjoint from L.

Let u_j be a sequence of J_{t_j}-holomorphic curves in the homology class $H - E_i$ in X. By SFT compactness, there is a Gromov-Hofer convergent subsequence u_{j_k} whose limit is a building F. We note some properties of the V-part F^V of such a building:

Lemma 5.4.12. *F^V is non-empty and falls into one of three categories ⓐ, ⓑ and ⓒ:*



where 

stands for a component which is an n-fold cover of a curve with $c_1^\Phi = A$.

Proof. The maximum principle says that W contains no closed holomorphic curves, hence F^V is nonempty. Arguing as in the proof of lemma 5.4.3, the relative first Chern class $c_1^\Phi(F^V)$ is 2 and F^V has at most two connected components. Since 1 is the minimal Chern number, multiple covering can only occur if the limit is a double cover of a simple curve with $c_1^\Phi = 1$. These are precisely the cases listed in the lemma. $\square$

Since there is a countable set of possible topologies and homology classes for simple genus 0, curves with $c_1^\Phi = 1,\ 2$, it can be ensured as in remarks 5.4.4 and 5.4.5 that all such moduli spaces are smooth and of the expected dimension. The dimension formula for classes with $c_1^\Phi = 2$ becomes

$$\begin{aligned} \dim(\mathcal{M}^A_{0,s^-}(J_\infty|_V)) &= 2 + 2s^- - 2\sum_{i=1}^{s^-} \operatorname{cov}(\gamma_i) \\ &\leq 2 \end{aligned}$$

which is only positive if

1. all asymptotic Reeb orbits are simple, in which case the expected dimension is 2, or
2. all but one of the asymptotic Reeb orbits are simple, the one exception being a doubly-wrapped orbit. The expected dimension in this case is 0.

This gives the following breakdown of the three cases from lemma 5.4.12:

Dimension 0:	ⓐ:	one double orbit
	ⓑ:	all simple orbits
	ⓒ:	all simple orbits
Dimension 2:	ⓐ:	all simple orbits

where we understand a multiply-covered curve to live "in a 0-dimensional moduli space" if its underlying simple curve has a moduli space of dimension 0. We call curves of type ⓐ with all simple orbits *good curves* and all other curves *bad curves.*

Let J_{t_j} be a sequence for which the curves $E_k(J_{t_j})$ and $S_{k\ell}(J_{t_j})$ Gromov-Hofer converge to J_∞-holomorphic limit buildings for all i, k, ℓ.

Lemma 5.4.13. *There exists a good curve in V which occurs as the V-part of the Gromov-Hofer limit of a sequence of J_{t_j}-holomorphic curves in X homologous to $H - E_i$ (for $i = 3, 4$).*

Proof. There is a dense set of points in V which do not lie on a bad curve, since there are countably many bad curves. Pick such a point, x. For every t_j there is a J_{t_j}-holomorphic curve homologous to $H - E_i$ passing through x. A subsequence of these Gromov-Hofer converges to a building in X^k_∞ through x, whose V-part must then be a good curve. □

Once we have this Gromov-Hofer limit, C_x, whose V-part is a good curve, we examine its asymptotic Reeb orbits. These are simple by construction and by the same argument as in lemma 5.4.6 the parts of C_x which land in the symplectisation of M are just Reeb cylinders and the components of C_x^W are α- and β-planes.

Lemma 5.4.14. *C_x^W is empty for generic neck-stretching data.*

Proof. Suppose that C_x^W were non-empty. Since $H - E_i$ has homological intersection 0 with E_1 and E_2, and since $E^W_{1,\infty}$ and $E^W_{2,\infty}$ consist of a single α- and a single β-plane each asymptotic to the same Reeb orbit R, the components of C_x^W must also be asymptotic to R in order to avoid intersections. Because $H - E_i$ has zero intersection with L, there must be at least one of each type of plane (recall that C_x^W consists of α- and β-planes), so C_x^V has $k \geq 2$ punctures. The evaluation map from the moduli space of C_x^V to ρ^k is transverse to all strata of the multi-diagonal for generic neck-stretching data, which have codimension at least 2 in ρ^k. Since the moduli space of C_x^V has dimension 2, the dimension of ρ^k is $2k$ and $\mathrm{ev}(C_x^V) = (R, \ldots, R)$ this means that $k = 2$ and nearby curves C' in the moduli of C_x^V have at least one asymptotic orbit not equal to R.

We must show that there is such a nearby curve C' which occurs as the V-part of a Gromov-Hofer limit C'_∞ of a sequence of J_{t_j}-holomorphic curves in X homologous to $H - E_i$. For then, since the asymptotic orbits of C' involve an orbit not equal to R, the W-part of C'_∞ contains a β- or α-plane not asymptotic to R which therefore intersects either $E^W_{1,\infty}$ or $E^W_{2,\infty}$ respectively, contradicting positivity of intersections and the fact that $E_1 \cdot (H - E_i) = 0$ and $E_2 \cdot (H - E_i) = 0$.

Lemma 5.4.15. *There is a nearby curve C' in the moduli of C_x^V which occurs as the V-part of the Gromov-Hofer limit of a sequence of $J_{t_{j_k}}$-holomorphic curves in X homologous to $H - E_i$ for a subsequence $j_k \subset j$.*

Proof. Let x_m be a sequence of points tending to x. For x_1 there is a subsequence $j_k^1 \subset j$ such that the $J_{t_{j_k^1}}$-holomorphic curves in X homologous to $H - E_i$ passing through x_1 Gromov-Hofer converge. Similarly, extract a subsequence $j_k^2 \subset j_k^1$ of this for curves passing through x_2 and (after iteratively constructing a subsequence j_k^m for each m) extract a Cantor diagonal subsequence $\delta_m = j_m^m$ for which all sequences of J_{δ_m}-holomorphic curves homologous to $H - E_i$ and passing through some fixed x_m Gromov-Hofer converge to a building A_m. Let C_m denote the V-part of A_m. We must show that as $m \to \infty$ there is a subsequence of these buildings which Gromov-Hofer converge to C_x^V, for then we may take $C' = C_M$ for large M.

There is certainly a Gromov-Hofer convergent subsequence C_{m_k} with limit C_{m_∞} whose V-part is $C^V_{m_\infty}$. Since C_x^V and $C^V_{m_\infty}$ intersect positively at x, they must be geometrically indistinct, for otherwise there would be a large k for which the J_{δ_k}-holomorphic curve homologous to $H - E_i$ through x_k and the curve through x would intersect with positive local intersection number, contradicting the fact that $(H - E_i) \cdot (H - E_i) = 0$.

Since x does not lie on a bad curve, the asymptotics of $C^V_{m_\infty}$ are all simple Reeb orbits and the symplectisation parts of C_{m_∞} are all Reeb cylinders. Since none of the curves in question had any marked points, such cylinders would necessarily be unstable, so cannot occur. Hence C_{m_k} Gromov-Hofer converges to C_x and for large k, C_{m_k} lives in the same moduli space as C_x. □

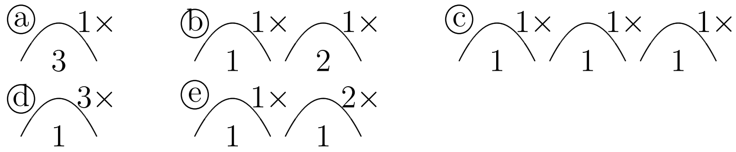
This proves the lemma.

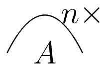
□

5.4.16 The class H

We finally turn our attention to Gromov-Hofer limits F of stable curves in the homology class H under neck-stretching.

Lemma 5.4.17. *F^V is non-empty and falls into one of five categories:*



where 

stands for a component which is an n-fold cover of curve with $c_1^\Phi = A$.

Proof. Non-emptiness follows from the maximum principle. The possible degenerations are a simple consequence of the fact that $c_1^\Phi(F^V) = 3$. $\square$

We begin with a sequence J_{t_j} for which the curves $E_1(J_{t_j})$, $E_2(J_{t_j})$, $S_{12}(J_{t_j})$ and $C_x^i(J_{t_j})$ Gromov-Hofer converge to J_∞-holomorphic buildings $E_{1,\infty}$, $E_{2,\infty}$, $S_{12,\infty}$ and C_∞^i (for $i = 3, 4$) such that $C_\infty^{i,W} = \emptyset$. Here $C_x^i(J_{t_j})$ are the unique J_{t_j}-holomorphic curves in the homology classes $H - E_i$ ($i = 3, 4$) passing through the point x. The aim is to find a subsequence $J_{t_{j_k}}$, a point $z \in V$ and a J_{t_j}-complex line in $T_z\overline{X}$ such that:

- $z \notin \Xi(J_{t_{j_k}})$ for any k (where $\Xi(J)$ is the union of the exceptional spheres $E_i(J)$),
- the sequence $H_z^\ell(J_{t_{j_k}})$ of stable $J_{t_{j_k}}$-holomorphic curves through z tangent to ℓ Gromov-Hofer converge to a J_∞-holomorphic building H_∞ whose W-part is empty.

Note that it makes sense to talk about ℓ being a J_{t_j}-complex line for all j, as the neck-stretch only affects the complex structure on the neck so $J_t|_V = J_0|_V$.

Lemma 5.4.18. *Suppose H_∞ is the Gromov-Hofer limit of a convergent sequence of stable $J_{t_{j_k}}$-holomorphic curves $H_z^\ell(J_{t_{j_k}})$ through z and tangent to ℓ. To prove that the W-part H_∞^W is empty it suffices to prove that H_∞^V is connected with simple asymptotic orbits distinct from R.*

Proof. By a similar argument to proposition 5.4.7 H_∞^W (if it were non-empty) would consist of α- and β-planes which would intersect $E_{1,\infty}$ or $E_{2,\infty}$ and thereby contradict the fact that $H \cdot E_1 = H \cdot E_2 = 0$. $\square$

Remark 5.4.19. *There is a Baire set of neck-stretching data such that the following is true. If H_∞ is a J_∞-holomorphic building obtained as the Gromov-Hofer limit of a sequence of J_{t_j}-holomorphic curves in the homology class H and H_∞^V is connected with simple Reeb asymptotics then any nearby H'_∞ in the moduli space of H_∞^V has an asymptotic Reeb orbit different from R.*

Lemma 5.4.20. *For a Baire set of neck-stretching data there exists a J_1-complex line in the tangent space at x and a subsequence $J_{t_{j_k}}$ such that the sequence of J_{t_k}-holomorphic curves $H^\ell_x(J_{t_{j_k}})$ Gromov-Hofer converges to a curve H_∞ such that H^V_∞ is connected and has simple Reeb asymptotics distinct from R.*

To prove the lemma, we will begin by naively finding a Gromov-Hofer sequence whose limit is connected with simple Reeb asymptotics, then we will use the remark above to find a nearby H'_∞ which has asymptotics distinct from R and finally observe that this punctured curve arises as the V-part of a Gromov-Hofer limit of curves as described by the lemma.

Proof. As in remark 5.4.4, J_1 can be chosen generically so that all moduli spaces of simple finite-energy punctured curves in V are regular and of the expected dimension. The index formula for curves with $c_1^\Phi = 3$ (such as the V-parts of our limits) is

$$\begin{aligned} \dim(\mathcal{M}^A_{0,s^-}(J_\infty|_V)) &= 4 + 2s^- - 2\sum_{i=1}^{s^-} \mathrm{cov}(\gamma_i) \\ &\leq 4 \end{aligned}$$

which is only nonnegative if:

1. all asymptotic Reeb orbits are simple, or
2. all but one of the asymptotic Reeb orbits are simple, the one exception being a doubly-wrapped orbit.
3. all but two of the asymptotic Reeb orbits are simple, the two exceptions being either: one simple and one triply-wrapped orbit or two doubly-wrapped orbits.

This gives us the following possible breakdown of the cases from lemma 5.4.17:

Dimension 0:	ⓐ:	one triple orbit
		two double orbits
	ⓑ:	one double orbit for component with $c_1^\Phi = 2$
	ⓒ:	all simple orbits
	ⓓ:	all simple orbits
	ⓔ:	all simple orbits
Dimension 2:	ⓐ:	one double orbit
	ⓑ:	all simple orbits
Dimension 4:	ⓐ:	all simple orbits

where we understand a multiply-covered curve to live "in a 0-dimensional moduli space" if its underlying simple curve has a moduli space of dimension 0. We call curves of type ⓐ with all orbits simple *good curves* and all other curves *bad curves*. Curves from the first five rows are called *very bad curves*. For a fixed almost complex structure $J_\infty|_V$, there is a dense set of points in V which do not lie on a very bad curve. Let $\Omega \subset V$ be a countable, dense set of such points. Let $\Lambda \subset \mathbb{P}^{J_1}(TV|_\Omega)$ be a set of J_1-complex lines such that $\Lambda \cap \mathbb{P}^{J_1}(T_xV)$ is dense in $\mathbb{P}^{J_1}(T_xV)$ for every $x \in \Omega$ (and recall that $J_1(x) = J_t(x)$ for all $t \in [1, \infty]$).

Number the elements of $\Omega = \{x_1, x_2, \dots\}$. There is a subsequence $t^1_{j_k} \subset t_j$ such that $x_1 \notin \Xi(J_{t^1_{j_k}})$ for all k, or else the Gromov-Hofer limit of stable curves through x_1 homologous to H would have a component with $c_1^\Phi = 1$ passing through x_1. Similarly there is a subsequence $t^2_{j_{k_m}} \subset t^1_{j_k}$ for which x_2 is not in $\Xi(J_{t^2_{j_{k_m}}})$ for any m. Iteratively construct a Cantor diagonal subsequence (written t_j for brevity) for which $x_i \notin \Xi(J_{t_j})$ for all i and j.

Proposition 4.3.4 now implies that for each $\ell \in \Lambda$ there is a sequence $H^\ell_x(J_{t_j})$ of J_{t_j}-holomorphic spheres in the homology class H passing through x and tangent to ℓ. Passing to a Cantor diagonal subsequence we can ensure that these all Gromov-Hofer converge to J_∞-holomorphic buildings $H^\ell_{x,\infty}$.

These buildings cannot all be bad curves, thanks to the following observation:

Lemma 5.4.21. *In $\mathbb{P}^{J_1}(TV)$ the set of points which are complex tangent lines to bad curves have open, dense complement.*

The lemma follows from the dimension formula: the set of such points form a countable union of 2- and 4-dimensional subspaces in the six-dimensional space $\mathbb{P}^{J_1}(TV)$. This subset is closed by SFT compactness.

Once we have this good curve, if it has an asymptotic orbit different from R then we have proved the lemma, setting $H_\infty = H^\ell_{x,\infty}$. If not, remark 5.4.19 implies that any nearby curve in the local moduli space of $H^\ell_{x,\infty}$ does have an asymptotic orbit different from R. Pick a sequence $\ell_i \in \Lambda \cap \mathbb{P}^{J_1}(T_xV)$ of good curves tending to ℓ. By the same argument as proved claim 5.4.15 we deduce that for large i we may take $H_\infty = H^{\ell_i}_{x,\infty}$. □

This completes the proof of proposition 5.4.1.

5.5 Analysis of limit-buildings: ternary case

Let L be a Lagrangian sphere in the ternary homology class $H - E_1 - E_2 - E_3$ in $\mathbb{D}_3$ or $\mathbb{D}_4$. The details of the analysis here are very similar to those in the previous section. The results are that for generic neck-stretching data:

- All exceptional classes E_1, E_2, E_3, (S_{12}, S_{13}, S_{23}) have limit-buildings consisting of a single α- (respectively β-) plane and S_{14} and E_4 have limit-buildings with empty W-parts.

- In $\mathbb{D}_3$, one can find a point $x \in V$ for which the three sequences of J_t-holomorphic curves homologous to $H - E_1$, $H - E_2$ and $H - E_3$ passing through x Gromov-Hofer converge to buildings with empty W-part.

- In $\mathbb{D}_4$, one can find a point x in $S_{14,\infty}$ for which the J_t-holomorphic curves homologous to $H - E_2$ and $H - E_3$ passing through x have Gromov-Hofer limits with empty W-part.

5.6 Proof of theorem 9

Let (X, ω) be a monotone symplectic Del Pezzo surface $\mathbb{D}_n$ $(n \leq 4)$, L an embedded Lagrangian sphere, J_1 a generic choice of neck-stretching data and $\{J_t\}_{t=1}^{T}$ be the corresponding family of ω-compatible complex structures obtained by stretching the neck around L. Extend J_t to a family $\{J_t\}_{t=0}^{T}$ where J_0 is the standard complex structure on $\mathbb{D}_n$ coming from thinking of it as a $n-1$-point complex blow-up at (∞, ∞) of $\mathbb{CP}^1 \times \mathbb{CP}^1$ with its product complex structure.

I will discuss the case of binary Lagrangian spheres in $n = 2$. Generalisation of the argument to binary and ternary spheres in $n = 3, 4$ should be clear.

Fix a point x and a complex direction ℓ at x such that for a suitable subsequence t_j the corresponding sequence $H_{x,\ell}(J_{t_j})$ of J_{t_j}-holomorphic curves in the homology class H through x tangent to ℓ Gromov-Hofer converge to a J_∞-holomorphic curve disjoint from L. The sequence $S_{12}(J_{t_j})$ also has a Gromov-Hofer convergent subsequence, $j_k \subset j$, whose limit curve is disjoint from L. Therefore for T large enough, $S_{12}(J_T)$ and $H_{x,\ell}(J_T)$ are disjoint from L.

Extend the family $\{J_t\}_{t=1}^{T}$ to a family $\{J_t\}_{t=0}^{T}$ where J_0 is an integrable complex structure coming from considering $\mathbb{D}_2$ as the blow-up of $\mathbb{CP}^2$ at two points in general position. For all $t \in [0, T]$ there are unique smooth, embedded J_t-spheres in the homology classes E_1, E_2 and S_{12} and the implicit function theorem implies that as t varies, these exceptional spheres undergo smooth isotopy (see [28], remark 3.2.8).

For each t the space $\mathcal{R}_t$ of pairs $\{(x', \ell') \in V \times \mathbb{P}^{J_t}(T_xV)\}$ for which there is a smooth J_t-curve homologous to H through x' tangent to ℓ' is open, dense and connected in the total space of $\mathbb{P}^{J_t}(TV)$: its complement consists of strata with codimension at least 2. Such curves are regular, so again by remark 3.2.8 in [28] there is a path in $\bigcup_{t\in[0,T]} \mathcal{R}_t$ which ends at (x, ℓ, T) and starts in $\mathcal{R}_0$. This gives a smooth isotopy of (smooth, regular, embedded) J_t-holomorphic curves H_t homologous to H, ending with one which is disjoint from L.

Now for each t consider the configuration $H_t \cup E_1(J_t) \cup E_2(J_t) \cup S_{12}(J_t)$. Choose families of Darboux balls centred at the points of S_{12} where the curves in the configuration intersect. In each family of balls, perform local perturbations of H_t, $E_1(J_t)$ and $E_2(J_t)$ to obtain a configuration of symplectic surfaces which intersect symplectically orthogonally. This perturbation is chosen to leave H_0, $E_1(J_0)$ and $E_2(J_0)$ unchanged and the Darboux balls can be chosen so that at $t = T$ they are disjoint from L (since L is disjoint from S_{12}). Write C_t for the resulting smooth isotopy of the configuration of perturbed spheres.

Each sphere has a symplectic neighbourhood to which the isotopy extends, by the symplectic neighbourhood theorem, and since the spheres are now symplectically orthogonal the isotopy extends over a neighbourhood of the whole configuration.

Now the existence of a global symplectomorphism $\Psi_t : X \to X$ for which $\Psi_t(C_0) = C_t$ follows from Banyaga's symplectic isotopy extension theorem (theorem 5) because these exceptional spheres generate $H_2(X, \mathbb{Z})$:

Therefore $L_t = \Psi_t^{-1}(L)$ is an isotopy of L through Lagrangian spheres which disjoins it from $S_{12}(J_0) = \Psi_T^{-1}(S_{12}(J_T))$ and from the J_0-holomorphic line $\Psi_T^{-1}(H_{x,\ell}(J_t))$. This proves theorem 9 for $\mathbb{D}_2$.

Chapter 6

The symplectomorphism group

6.1 Results

In this section, we use known results on symplectomorphism groups of affine varieties to calculate the weak homotopy type of the symplectomorphism groups of some Del Pezzo surfaces. We prove the following.

Theorem 3. *Let $Symp_0(X)$ denote the group of symplectomorphisms of X acting trivially on homology and let $Diff^+(S^2, 5)$ denote the group of diffeomorphisms of S^2 fixing five points.*

- $Symp(\mathbb{D}_3) \simeq T^2$,
- $Symp(\mathbb{D}_4) \simeq \star$,
- $Symp(\mathbb{D}_5) \simeq Diff^+(S^2, 5)$.

where all $\simeq$ signs denote weak homotopy equivalence.

The third result in the theorem deserves some comment. Seidel has shown [37] that there are maps $C/G \to B\mathrm{Symp}(\mathbb{D}_5) \to C/G$ whose composition is homotopic to the identity. Here C is the configuration space of five ordered points on S^2 and G is the group $\mathbb{P}SL_2\mathbb{C}$ of Möbius transformations of S^2 (the map $C/G \to B\mathrm{Symp}(\mathbb{D}_5)$ was defined in section 1.3.4). G acts freely on C, so we get a principal G-bundle:

$$\begin{array}{ccc} G & \xrightarrow{a} & C \\ & & \downarrow \\ & & C/G \end{array}$$

The group $\mathrm{Diff}^+(S^2)$ also acts on C and the orbit-stabiliser theorem yields a fibration

$$\begin{array}{ccc} \mathrm{Diff}^+(S^2,5) & \longrightarrow & \mathrm{Diff}^+(S^2) \\ & & \downarrow{\scriptstyle b} \\ & & C \end{array}$$

The maps a and b are actually homotopy equivalent (in view of the fact that the inclusion $G \to \mathrm{Diff}^+(S^2)$ is a homotopy equivalence), so C/G is weakly homotopy equivalent to $B\mathrm{Diff}^+(S^2,5)$. In fact, Teichmüller theory shows that $\mathrm{Diff}^+(S^2,5)$ is weakly homotopy equivalent to its group B of components. We can find the group B as follows. $\pi_1(C)$ is the group $P\mathrm{Br}(S^2,5)$ of five-strand pure braids on the sphere. $\pi_1(G)$ is $\mathbb{Z}/2$, generated by a loop of rotations through 0 to 2π. By the homotopy long exact sequence associated to the fibration $G \to C \to C/G$, $\pi_1(C/G)$ is the cokernel of the map $\mathbb{Z}/2 \to P\mathrm{Br}(S^2,5)$. It is not hard to see that the image of this map is the full twist τ (which has order 2 in the braid group), so $B \cong P\mathrm{Br}(S^2,5)/\langle\tau\rangle$.

6.2 Outline of proof

The proofs in all three cases run along similar lines. We outline the general picture before filling in details individually. Therefore let (X,ω) denote any of $\mathbb{D}_3$, $\mathbb{D}_4$ or $\mathbb{D}_5$, let $\mathcal{J}$ denote the space of ω-tame almost complex structures on X and let $\mathrm{Symp}_0(X)$ denote the group of symplectomorphisms acting trivially on $H^*(X,\mathbb{Z})$.

In each case we will identify a divisor $C = \bigcup_i C_i \subset X$ such that

- C consists of embedded J-holomorphic -1-spheres which intersect one another symplectically orthogonally,
- $H^2(X,C;\mathbb{R}) = 0$.

Definition 6.2.1. *A* standard configuration *in X will mean a configuration $S = \bigcup_i S_i$ of embedded symplectic spheres such that*

- *$[S_i] = [C_i]$ for all i,*
- *there exists a $J \in \mathcal{J}$ simultaneously making every component S_i into a J-holomorphic sphere,*
- *at every intersection point of the configuration, the components intersect ω-orthogonally.*

Let $\mathcal{C}_0$ denote the space of standard configurations.

Proposition 6.2.2. *$\mathcal{C}_0$ is weakly contractible.*

Proof. The proof of this proposition is very similar to the proof of proposition 3.2.3. The input from Gromov's theory of pseudoholomorphic curves is theorem 7, mentioned in the proof of proposition 3.3.7. $\square$

In particular, any $S_1, S_2 \in \mathcal{C}_0$ are isotopic through standard configurations. The property that the configurations are symplectically orthogonal where they intersect and the condition $H^2(X, C; \mathbb{R})$ allow us to extend such an isotopy to a global symplectomorphism of X (by theorem 5). Therefore:

Proposition 6.2.3. *$Symp_0(X)$ acts transitively on $\mathcal{C}_0$.*

The orbit-stabiliser theorem now gives us a fibration:

$$\begin{array}{ccc} \mathrm{Stab}(C) & \longrightarrow & \mathrm{Symp}_0(X) \\ & & \downarrow \\ & & \mathcal{C}_0 \end{array}$$

so the group $\mathrm{Stab}(C)$ of symplectomorphisms of X which fix the configuration C setwise (and act trivially on the set of components) is weakly homotopy equivalent to $\mathrm{Symp}_0(X)$ (by the homotopy long exact sequence, since $\mathcal{C}_0$ is weakly contractible).

The next step is to investigate $\mathrm{Stab}(C)$. An element $\phi \in \mathrm{Stab}(C)$ restricts to give an element $(\phi|_{C_1}, \ldots, \phi|_{C_n}) \in \mathrm{Symp}(C)$ (see section 2.3.3). The orbit-stabiliser theorem implies that this restriction map fits into a fibration:

$$\begin{array}{ccc} \mathrm{Stab}^0(C) & \longrightarrow & \mathrm{Stab}(C) \\ & & \downarrow \\ & & \mathrm{Symp}(C) \end{array} \tag{*}$$

where $\mathrm{Stab}^0(C)$ is the group of symplectomorphisms of X which fix C pointwise.

As explained in section 2.3.2, the group $\mathrm{Symp}(C)$ can be understood purely in terms of $\mathrm{Diff}^+(C)$, thanks to Moser's theorem, and thereby reduced to a problem in Teichmüller theory. To understand the stabiliser $\mathrm{Stab}^0(C)$, we need to consider the fibration:

$$\begin{array}{ccc} \mathrm{Stab}^1(C) & \longrightarrow & \mathrm{Stab}^0(C) \\ & & \downarrow \\ & & \mathcal{G} \end{array}$$

which takes a symplectomorphism fixing C to the induced map on the normal bundles of C (see section 2.3.1). Here $\mathcal{G}$ is the group of symplectic gauge transformations of the normal bundles to components of C and $\mathrm{Stab}^1(C)$ consists of those symplectomorphisms fixing C pointwise and acting trivially on the normal bundles of its components. By the symplectic neighbourhood theorem:

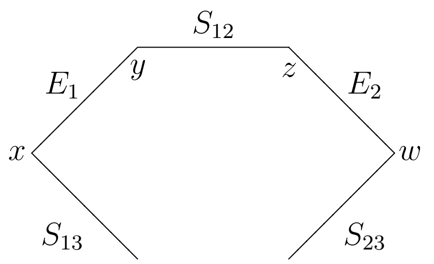
Lemma 6.2.4. *$Stab^1(C)$ is weakly homotopy equivalent to the group of compactly supported symplectomorphisms of $U = X \setminus C$.*

In all our cases we will understand $\mathrm{Symp}_c(U)$ and $\mathcal{G}$. The homotopy exact sequences of the various fibrations (pictured below) will allow us to work backward, finding the homotopy groups of $\mathrm{Stab}^0(C)$ and $\mathrm{Stab}(C) \simeq \mathrm{Symp}_0(X)$.

$$\begin{tikzcd}
\mathrm{Symp}_c(U) \arrow[r] & \mathrm{Stab}^0(C) \arrow[r] \arrow[d] & \mathrm{Stab}(C) \arrow[r, "\simeq"] \arrow[d] & \mathrm{Symp}_0(X) \arrow[d] \\
 & \mathcal{G} & \mathrm{Symp}(C) & \mathrm{Orb}(C) \simeq \star
\end{tikzcd}$$

6.3 $\mathbb{D}_3$

Here we consider the configuration of exceptional curves in the homology classes $S_{12} = H - E_1 - E_2$, $S_{13} = H - E_1 - E_3$, $S_{23} = H - E_2 - E_3$, E_1 and E_2. If we included E_3, these would form a hexagonal configuration, but in order to ensure $H^2(X, C; \mathbb{R}) = 0$, we need to omit E_3.



The complement of $U = X \setminus C$ is biholomorphic to $\mathbb{C} \times \mathbb{C}^*$ and ω restricted to U is $-dd^c h$ for a plurisubharmonic function h on U. By proposition 2.2.4, $\mathrm{Symp}_c(U) \simeq \mathrm{Symp}_c(\mathbb{C} \times \mathbb{C}^*)$. By theorem C, $\mathrm{Symp}_c(\mathbb{C} \times \mathbb{C}^*)$ is contractible, so $\mathrm{Stab}^0(C) \simeq \mathcal{G}$. Let us calculate the groups $\mathcal{G}$ and $\mathrm{Symp}(C)$:

- $\mathcal{G}(S_{13}) \cong \mathcal{G}(S_{23}) \cong \mathcal{G}_1 \simeq \star$,
- $\mathcal{D}(S_{13}) \cong \mathcal{D}(S_{23}) \cong \mathcal{D}_1 \simeq S^1$,
- $\mathcal{G}(S_{12}) \cong \mathcal{G}(E_1) \cong \mathcal{G}(E_2) \cong \mathcal{G}_2 \simeq \mathbb{Z}$,
- $\mathcal{D}(S_{12}) \cong \mathcal{D}(E_1) \cong \mathcal{D}(E_2) \cong \mathcal{D}_2 \simeq S^1$,

therefore $\mathcal{G} \simeq \mathbb{Z}^3$ and $\mathrm{Symp}(C) \simeq (S^1)^5$. Since $\mathrm{Stab}^0(C) \simeq \mathcal{G}$ and $\mathrm{Stab}(C) \simeq \mathrm{Symp}(\mathbb{D}_3)$, the fibration (*) yields the long exact sequence:

$$1 \to \pi_1(\mathrm{Symp}(\mathbb{D}_3)) \to \mathbb{Z}^5 \to \mathbb{Z}^3 \to \pi_0(\mathrm{Symp}(\mathbb{D}_3)) \to 1$$

The calculation reduces to understanding the map $\mathbb{Z}^5 \to \mathbb{Z}^3$. This comes from a map $\pi_1(\mathrm{Symp}(C)) \to \pi_0(\mathcal{G})$, so this can actually be understood purely by considering a neighbourhood of C as explained in section 2.3.3. Let us pick a basis for $\pi_1(\mathrm{Symp}(C))$. Let $\mathrm{rot}(C_j)$ denote the element represented by the loops $\phi_{2\pi}$ defined in lemma 2.3.4, i.e. the loop of sympletomorphisms which rotate C_j through 2π and leave the other components fixed. These generate the group $\pi_1(\mathrm{Symp}(C))$.

By lemma 2.3.4, the map $\mathbb{Z}^5 \to \mathbb{Z}^3$ is:

$$\begin{array}{rcll} \mathrm{rot}(S_{13}) & \mapsto & g_{E_1}(x) & \in \pi_0(\mathcal{G}(E_1)) \\ \mathrm{rot}(S_{23}) & \mapsto & g_{E_2}(w) & \in \pi_0(\mathcal{G}(E_2)) \end{array}$$

and

$$\begin{array}{rcll} \mathrm{rot}(E_1) & \mapsto & (0, g_{S_{13}}(x)) & \in \pi_0(\mathcal{G}(S_{13})) \times \pi_0(\mathcal{G}(S_{12}) \\ \mathrm{rot}(E_2) & \mapsto & (g_{S_{12}}(z), 0) & \in \pi_0(\mathcal{G}(S_{12})) \times \pi_0(\mathcal{G}(S_{23})) \\ \mathrm{rot}(S_{12}) & \mapsto & (g_{E_1}(y), g_{E_2}(z)) & \in \pi_0(\mathcal{G}(E_1)) \times \pi_0(\mathcal{G}(E_2)) \end{array}$$

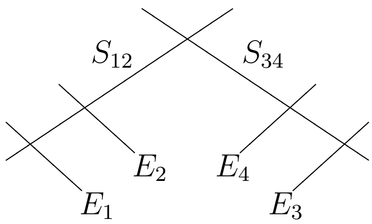
Therefore the map is surjective with kernel of rank 2. This implies

$$\pi_1(\mathrm{Symp}_0(\mathbb{D}_3)) = \mathbb{Z}^2, \ \pi_0(\mathrm{Symp}_0(\mathbb{D}_3)) = 0$$

as stated.

6.4 $\mathbb{D}_4$

The configuration C consists of exceptional spheres in the homology classes $S_{12} = H - E_1 - E_2$, $S_{34} = H - E_3 - E_4$, E_1, E_2, E_3 and E_4.



Let us calculate the homotopy types of $\mathcal{G}$ and $\mathrm{Symp}(C)$:

- $\mathcal{G}(S_{12}) \cong \mathcal{G}(S_{34}) \cong \mathcal{G}_3 \simeq \mathbb{Z}^2$,
- $\mathcal{D}(S_{12}) = \mathcal{D}(S_{34}) = \mathcal{D}_3 \simeq \star$,
- $\mathcal{G}(E_1) \cong \cdots \cong \mathcal{G}(E_4) \cong \mathcal{G}_1 \simeq \star$,
- $\mathcal{D}(E_1) \cong \cdots \cong \mathcal{D}(E_4) = \mathcal{D}_1 \simeq S^1$,

Therefore $\mathcal{G} \simeq \mathbb{Z}^4$ and $\mathrm{Symp}(C) \simeq (S^1)^4$.

The complement $U = X \setminus C$ is biholomorphic to $\mathbb{C}^2$ and the restriction of ω to U arises as $-dd^c h$ for a plurisubharmonic function h on U, so by proposition 2.2.4 $\mathrm{Symp}_c(U) \simeq \mathrm{Symp}_c(\mathbb{C}^2)$. Gromov has shown $\mathrm{Symp}_c(\mathbb{C}^2) \simeq \star$. This simplifies the fibrations:

$$\begin{array}{ccccc} \star & \longrightarrow & \mathrm{Stab}^0(C) & \longrightarrow & \mathrm{Stab}(C) \simeq \mathrm{Symp}_0(\mathbb{D}_4) \\ & & \downarrow \simeq & & \downarrow \\ & & \mathcal{G} \simeq \mathbb{Z}^4 & & \mathrm{Symp}(C) \simeq (S^1)^4 \end{array}$$

We therefore get a long exact sequences of homotopy groups:

$$1 \to \pi_1(\mathrm{Symp}_0(\mathbb{D}_4)) \to \mathbb{Z}^4 \to \mathbb{Z}^4 \to \pi_0(\mathrm{Symp}_0(\mathbb{D}_4)) \to 1$$

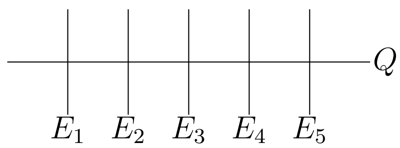
In the notation introduced in the previous section, the map $\mathbb{Z}^4 \to \mathbb{Z}^4$ is given by:

$$\mathrm{rot}(E_i, x_i) \mapsto g_{S_{ik}}(x_i) \in \pi_0(\mathcal{G}(S_{ik}))$$

so it is an isomorphism and all the homotopy groups of $\mathrm{Symp}_0(\mathbb{D}_4)$ vanish.

6.5 $\mathbb{D}_5$

The relevant configuration C is the total transform of the conic through the five blow-up points, i.e. the exceptional curves in homology classes $E_1, \ldots, E_5$ and $Q = 2H - \sum_{i=1}^{5} E_i$.



Let us calculate the homotopy types of $\mathcal{G}$ and $\mathrm{Symp}(C)$:

- $\mathcal{G}(Q) \cong \mathcal{G}_5 \simeq \mathbb{Z}^4$,
- $\mathcal{D}(Q) \cong \mathrm{Diff}(5, S^2)$
- $\mathcal{G}(E_1) \cong \cdots \cong \mathcal{G}(E_5) \cong \mathcal{G}_1 \simeq \star$,
- $\mathcal{D}(E_1) \cong \cdots \cong \mathcal{D}(E_5) \cong \mathcal{D}_1 \simeq S^1$,

so $\mathcal{G} \simeq \mathbb{Z}^4$ and $\mathrm{Symp}(C) \simeq \mathrm{Diff}(5, S^2) \times (S^1)^5$.

The complement $U = X \setminus C$ is biholomorphic to the complement of a conic in $\mathbb{CP}^2$ and the restriction of ω to U is $-dd^c h$ for a plurisubharmonic function h, so by proposition

2.2.4 $\mathrm{Symp}_c(U) \simeq \mathrm{Symp}_c(T^*\mathbb{RP}^2)$. By section B, $\mathrm{Symp}_c(T^*\mathbb{RP}^2) \simeq \mathbb{Z}$. The fibrations become:

$$\begin{array}{ccccc} \mathbb{Z} & \longrightarrow & \mathrm{Stab}^0(C) & \longrightarrow & \mathrm{Stab}(C) \simeq \mathrm{Symp}(\mathbb{D}_5) \\ & & \downarrow \simeq & & \downarrow \\ & & \mathcal{G} \simeq \mathbb{Z}^4 & & \mathrm{Symp}(C) \simeq \mathrm{Diff}(5, S^2) \times (S^1)^5 \end{array}$$

From this we deduce that $\mathrm{Stab}^0(C)$ is weakly equivalent to $\mathbb{Z}^5$ and that the relevant long exact sequence of homotopy groups is:

$$1 \to \pi_1(\mathrm{Symp}(\mathbb{D}_5)) \to \mathbb{Z}^5 \xrightarrow{\psi} \mathbb{Z}^5 \to \pi_0(\mathrm{Symp}(\mathbb{D}_5)) \to B \to 1$$

where B is the group $P\mathrm{Br}(5, S^2)/\langle\tau\rangle$ of components of $\mathrm{Diff}(5, S^2)$.

The key, then, is to understand this map $\psi : \mathbb{Z}^5 \to \mathbb{Z}^5$. The composition of this map with $\pi_0(\mathrm{Stab}^0(C)) \to \mathcal{G} \cong \mathbb{Z}^4$ is surjective by lemma 2.3.4 (the preimage of $g_Q(x_i) \in \pi_0(\mathcal{G}(Q))$ is $\mathrm{rot}(E_i, x_i)$). It remains to show that if χ is the isotopy class of the Dehn twist in the Lagrangian $\mathbb{RP}^2 \subset U$ then $\chi = \psi(\eta)$ for some $\eta \in \pi_1(\mathrm{Symp}(C))$.

We begin with a lemma whose proof is analogous to that of lemma 1.8 from [37] (which is the same statement for Lagrangian spheres):

Lemma 6.5.1. *Let $L \subset X$ be a Lagrangian $\mathbb{RP}^2$, $\mho$ a Weinstein neighbourhood of L and suppose there is a Hamiltonian circle action on $X \setminus L$ which commutes with the round cogeodesic flow on $\mho \setminus L$. Then the Dehn twist in L is symplectically isotopic to id.*

Let μ be the moment map for the $SO(3)$-action on $T^*\mathbb{RP}^2$. Then $||\mu||$ generates a Hamiltonian circle action on $T^*\mathbb{RP}^2 \setminus \mathbb{RP}^2$ which commutes with the round cogeodesic flow. Symplectically cutting along a level set of $||\mu||$ gives $\mathbb{CP}^2$ and the reduced locus is a conic. Pick five points on the conic and $||\mu||$-equivariant balls of equal volume centred on them. $\mathbb{D}_5$ is symplectomorphic to the blow up in these five balls and the circle action preserves the exceptional locus (the union of the five blow-up spheres and the proper transform of the conic). Hence we have constructed a loop in $\mathrm{Symp}(C)$ which maps to χ. Since the circle action simultaneously rotates the normal bundles to the blow-up spheres it should be clear that this element corresponds to the diagonal element $(1, \ldots, 1) \in \mathbb{Z}^5$.

This completes the proof of theorem D.

Appendix A

Postponed proofs

A.1 Proof of lemma 3.2.10

Let $S = \bigcup_{i=1}^{n} S_i$ be a union of embedded symplectic 2-spheres in a symplectic 4-manifold X. Suppose that the various components intersect transversely and that there are no triple intersections. Suppose further that there is an ω-compatible J for which all the components S_i are J-holomorphic. Let $\mathcal{H}(S)$ denote the space of ω-compatible almost complex structures J for which all components S_i are simultaneously J-holomorphic.

Lemma 4.2.10 *If all the intersection between components of S are symplectically orthogonal then the space $\mathcal{H}(S)$ is weakly contractible.*

Proof. Let $\mathcal{J}(S_i)$ denote the (contractible) space of $\omega|_{S_i}$-compatible almost complex structures on S_i and $\mathcal{J}(S)$ denote the product $\prod_{i=1}^{n} \mathcal{J}(S_i)$. The space $\mathcal{H}$ maps to $\mathcal{J}(S)$ by restriction. We want to show this is a homotopy equivalence.

We proceed in stages. First define $\mathcal{J}|_{S_i}$ to be the space of ω-compatible almost complex structures on $TX|_{S_i}$ and $\mathcal{J}|_S$ to be the product $\prod_{i=1}^{n} \mathcal{J}|_{S_i}$. The restriction map $\mathcal{H}(S) \to \mathcal{J}(S)$ factors through restriction maps:

$$\mathcal{H}(S) \to \mathcal{J}|_S \to \mathcal{J}(S)$$

We first analyse the map $\mathcal{J}|_S \to \mathcal{J}(S)$.

Lemma A.1.1. *$\mathcal{J}|_S \to \mathcal{J}(S)$ is a fibration.*

Proof. We illustrate the proof by proving path-lifting. Let ν_i denote the normal bundle to S_i, identified canonically as a subbundle of $TX|_{S_i}$ (the ω-orthogonal complement to

TS_i). Fix an $\omega|_{\nu_i}$-compatible almost complex structure k_i on ν_i. If

$$\gamma = (j_1(\cdot), \ldots, j_n(\cdot)) : I \to \mathcal{J}(S) = \prod_{i=1}^{n} \mathcal{J}(S_i)$$

is a path of complex structures on $S = S_1 \cup \cdots \cup S_n$ then $j_i \oplus k_i$ is an ω-compatible almost complex structure on $TX|_{S_i}$. Unfortunately, there is no guarantee that $j_a(t) \oplus k_a = k_b \oplus j_b(t)$ at the intersection point $x \in S_a \cap S_b$. To rectify this, fix a Darboux ball $\iota : (B, \omega_0) \to (X, \omega)$ centred at x for which (S_a) is identified under ι with $\Pi_a = B \cap \mathbb{R}^2 \times \{0\}$ and S_b with $\Pi_b = B \cap \{0\} \times \mathbb{R}^2$. Let η be a radial cut-off function on B which is equal to 1 on a neighbourhood of 0 and equal to 0 outside a small ball.

We now modify k_a and k_b so that they agree with $j_b(t)$ and $j_a(t)$ at x. Let c stand for either a or b. Let g_c be the metric on the normal bundle $\nu\Pi_c$ to Π_c corresponding to the pullback $\iota^* k_c$. Let $j(t)$ be the constant complex structure on B such that $\iota_* j(t) = j_a(t) \oplus j_b(t)$ at x. This $j(t)$ defines a metric $\tilde{g}_c(t)$ on the normal bundle to Π_c. Since the space of metrics is convex, define a metric

$$g_c'(t) = (1 - \eta) g_c + \eta \tilde{g}_c(t)$$

This in turn defines an endomorphism $A_c(t)$ of $\nu\Pi_c$ via $g_c'(t) = \omega_0(A_c(t)\cdot, \cdot)$. By the usual trick, $(A_c(t) A_c^{\dagger}(t))^{-1/2} A_c(t)$ defines an ω_0-compatible almost complex structure on $\nu\Pi_c$. Push this forward along ι. The new almost complex structures $k_a'(t)$ and $k_b'(t)$ agree with k_a and k_b outside the Darboux ball but now $k_a'(t) = j_b(t)$ and $k_b'(t) = j_a(t)$ at x.

Do this at every intersection point to get a 1-parameter family $k_i(t)$ of almost complex structures on ν_i. These give almost complex structures $j_i(t) \oplus k_i(t)$ on $TX|_{S_i}$ such that $j_a(t) \oplus k_a(t) = k_b(t) \oplus j_b(t)$ at any intersection point $x \in S_a \cap S_b$ and hence we get a well-defined lift of γ to a path in $\mathcal{J}|_S$. □

Lemma A.1.2. *The fibre of $\mathcal{J}|_S \to \mathcal{J}(S)$ is contractible.*

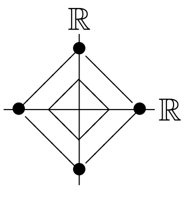
Proof. The fibre is the space of almost complex structures on $TX|_S$ which are fixed along the tangent directions to each S_i. For each S_a, let $\mathcal{G}(S_a)$ denote the group of symplectic gauge transformations of ν_a (the normal bundle to S_a) which equal the identity at intersection points $x \in S_a \cap S_b$. This acts transitively on the space of $\omega|_{\nu_a}$-compatible almost complex structures which agree with $j_a \oplus j_b$ at $x \in S_a \cap S_b$. Hence $\mathcal{G}(S) = \prod_{i=1}^{n} \mathcal{G}(S_i)$ acts transitively on the fibre of $\mathcal{J}|_S \to \mathcal{J}(S)$, making it into a homogeneous space $\mathcal{G}(S)/\mathrm{Stab}(J)$. Since the inclusion $U(1) \to \mathrm{Sp}(2)$ is a homotopy equivalence, $\mathcal{G}(S)$ is homotopy equivalent to the group of unitary gauge transformations, $\mathcal{G}_u(S)$, which is also the stabiliser of any given almost complex structure on the normal bundle. The orbit-stabiliser theorem implies that the homogeneous space $\mathcal{G}(S)/\mathcal{G}_u(S)$ is contractible. □

This shows that the space $\mathcal{J}|_S$ is weakly contractible, as it is the total space of a fibration over a contractible space with contractible fibres. We now analyse the restriction map $\mathcal{H}(S) \to \mathcal{J}|_S$.

Lemma A.1.3. *$\mathcal{H}(S) \to \mathcal{J}|_S$ is a fibration.*

Proof. By the symplectic neighbourhood theorem, each component of S_i has a neighbourhood M_i isomorphic to a neighbourhood ν_i of the zero section in its normal bundle via an isomorphism $\phi_i : \nu_i \to M_i$. Since the various components intersect symplectically orthogonally, these identifications can be chosen compatibly in the sense that $S_j \cap M_i$ is identified with a normal fibre in ν_i. Let $\gamma(t) = (J_1(t), \ldots, J_n(t))$ be a path in $\mathcal{J}|_S$. Each $\phi_i^* J_i(t)$ can be canonically extended to an ω-compatible almost complex structure on ν_i: $J_i(t)$ automatically defines an almost complex structure on the normal bundle to S_i, and the horizontal spaces ω-orthogonal to the normal fibres give a connection which allows one to lift $J_i(t)|_{TS_i}$ to an almost complex structure on the total space of ν_i.

This does not quite work at the intersection points where one must fix a Darboux chart and implant a local model. Suppose the two intersecting components are sent to the $\mathbb{R}^2 \times \{0\}$ and $\{0\} \times \mathbb{R}^2$ planes in the Darboux chart and the almost complex structure $J(p, q)$ is specified along these planes (i.e. for points $(p, 0)$ and $(0, q)$). Then, working with associated metrics, one can interpolate linearly as follows. Let $S^1(r) \times \{0\}$ and $\{0\} \times S^1(r)$ denote the unit circles (for the standard Euclidean metric) of the coordinate planes. For any point x of $\mathbb{R}^4$ there exists an r and a unique line segment connecting $S^1(r) \times \{0\}$ to $\{0\} \times S^1(r)$ containing x. Use the linear coordinate of this line to interpolate the associated metrics. The analogous picture in $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ is familiar, where the dots lie on the unit circles:



Modifying this near the boundary of the Darboux ball to agree with the J constructed from connections, we obtain an ω-compatible almost complex structure $\tilde{\gamma}(t)$ on a neighbourhood of S.

Fix an arbitrary ω-compatible almost complex structure K on X. Interpolating associated metrics using a cut-off function in a neighbourhood of S, one can extend $\tilde{\gamma}(t)$ over X so that it agrees with K outside a neighbourhood of S. This gives a lift of the path γ. The proof of the covering homotopy property is similar but cumbersome. □

Lemma A.1.4. *The fibre of $\mathcal{H}(S) \to \mathcal{J}|_S$ is contractible.*

Proof. This is standard. The fibre $\mathcal{F}$ consists of ω-compatible almost complex structures making S holomorphic which agree with a fixed almost complex structure J along S. We define a deformation retraction of $\mathcal{F}$ to $\{J\}$: for $J' \in \mathcal{F}$, define the metric $g_{J'} = \omega(\cdot, J'\cdot)$. The path $g_{J'}(t) = tg_J + (1-t)g_{J'}$ of metrics defines a path of almost complex structures connecting J' and J in the usual way. □

This shows that $\mathcal{H}(S)$ is weakly contractible, since it is the total space of a fibration over a weakly contractible space $\mathcal{J}|_S$ with contractible fibres. □

A.2 Proofs of transversality results

The proofs are closely modelled on [28], propositions 3.2.1 and 3.4.2.

Proof of proposition 5.2.23. To prove that the universal moduli space is a manifold it suffices (by the implicit function theorem for Banach manifolds) to show that the vertical differential

$$D\sigma(J,F) : W^{k,p}_{\delta_Z}(F^*T\overline{X}) \times \mathcal{C}^\ell(\mathrm{End}(T\overline{X}, J, \eta, \omega) \to W^{k-1,p}_{\delta_Z}(\Lambda^{0,1}_j \otimes_{(j,J)} F^*T\overline{X})$$

is surjective for generic δ_Z whenever F is a simple finite-energy J-curve. Notice that we have omitted the finite-dimensional V_Z and X_Z factors, as these do not affect the argument. This differential is given by

$$D\sigma(J,F)(Y,y,\xi) = D_F\xi + Y \circ dF \circ j$$

where $D_F\xi = \zeta ds + \overline{J}(F)\zeta dt$ in local isothermal coordinates on $S \setminus Z$ with $\zeta = \nabla_{\partial_s F}\xi + \overline{J}\nabla_{\partial_t}\xi + (\nabla_\xi \overline{J})\partial_t F$. Here ∇ is the $\omega(-, \overline{J}-)$ Levi-Civita connection. By [6], proposition 5.2, the operator D_F is Fredholm and hence its image is closed. To prove surjectivity it suffices to prove that the image of D_F is dense if F is a simple $\overline{J}$-holomorphic curve.

Case $\mathbf{k=1}$: If the image of D_F were not dense then by the Hahn-Banach theorem there would be a non-zero $\beta \in L^q_{-\delta_Z}(\Lambda^{0,1}_j \otimes_{(j,J)} F^*T\overline{X})$ with $q^{-1}+p^{-1}=1$ and

$$\int_{S\setminus Z} \langle \beta, D_F\xi \rangle \, d\mathrm{vol} = 0 \tag{A.1}$$

$$\int_{S\setminus Z} \langle \beta, Y \circ dF \circ j \rangle \, d\mathrm{vol} = 0 \tag{A.2}$$

The first of these three equations implies (via elliptic regularity) that β is of Sobolev class $W^{1,p}$ and that $D^*\beta = 0$. Aronszajn's unique continuation theorem means that β can therefore only vanish on a discrete set of points.

It follows from [39], corollaries 2.5 and 2.6 that a holomorphic curve which is simple has a finite number of non-injective points. We will show that β vanishes on any non-injective point and hence vanishes identically. Let $x_0 \in S \setminus Z$ be an injective point of F.

Lemma A.2.1. *β vanishes at x_0.*

If not, pick $Y_0 \in \mathrm{End}(T_{F(x_0)}\overline{X}, J_{F(x_0)}, \eta_{F(x_0)}, \omega_{F(x_0)})$ such that

$$\langle \beta_{x_0}, Y_0 \circ dF(x_0) \circ j(z_0) \rangle > 0$$

and let $Y \in \mathcal{C}^\ell(\mathrm{End}(T\overline{X}, \overline{J}, \eta, \omega)$ be such that $Y(F(x_0)) = Y_0$. This ensures that $\langle \beta, Y \circ dF \circ j \rangle$ is positive in some neighbourhood of x_0. Since x_0 is an injective point this contains the inverse image under F of a small ball U_0 centred at $F(x_0)$. If $C : \overline{X} \to [0,1]$ is a cut-off function supported in U_0 with $C(F(x_0)) = 1$ then

$$\langle \beta, CY \circ dF \circ j \rangle \geq 0$$

and is positive at x_0 so equation A.2 cannot hold for CY so $\beta(x_0) = 0$.

Case k $>$ 1: Follows by elliptic regularity. □

Before proving proposition 5.2.24 we prove the following lemma.

Lemma A.2.2. *Let $p > 2$, $J \in \mathcal{J}^\ell$, $w \in S \setminus Z$ and $F : S \setminus Z \to \overline{X}$ be a simple punctured finite-energy $\overline{J}$-holomorphic curve. For every $\epsilon > 0$ and tangent vector $v_Z \in T_{ev_R(J,F)}R_Z$ there is a vector field $\xi \in T_{(J,F)}\mathcal{B}^{\ell,p}_{\delta_Z}$ and a section $Y \in \mathcal{C}^\ell(End(T\overline{X}, J, \eta, \omega)$ such that*

$$D_F\xi + Y(F)dF \circ j = 0, \ \mathit{supp}(Y) \subset B_\epsilon(u(w))$$

and the X_Z-component of ξ is v_Z.

Proof of lemma A.2.2. Let π_X denote the projection of $T_F\mathcal{B}^{k,p}_{\delta_Z}$ to X_Z. Given a vector field $\xi' \in T_F\mathcal{B}^{k,p}_{\delta_Z}$ with $\pi_X(\xi') = v_Z$ we will show that there is a $\xi'' \in T_F\mathcal{B}^{k,p}_{\delta_Z}$ with $\pi_X(\xi'') = 0$ and a $Y \in \mathcal{C}^\ell(\mathrm{End}(T\overline{X}, J, \eta, \omega))$ supported in $B_\epsilon(F(w))$ such that

$$D_F(\xi' + \xi'') + Y \circ dF \circ j = 0$$

It suffices to show that the map

$$(\xi, Y) \mapsto D_F\xi + Y \circ dF \circ j$$

is surjective for Y supported in $B_\epsilon(F(w))$ and $\xi \in \pi_X^{-1}(0)$, for then we may take (ξ'', Y) in the preimage of $-D_F\xi'$. Let $\mathcal{Z}_k$ denote the image of this operator for ξ of regularity $W^{k,p}_{\delta_Z}$.

Lemma A.2.3. *$\mathcal{Z}^k = W^{k-1,p}_{\delta_Z}(\Lambda^{0,1}_j \otimes_{(j,J)} F^*T\overline{X})$ for $k \leq \ell$.*

This will follow by induction from elliptic regularity once we have proved that $\mathcal{Z}^1 = L^p_{\delta_Z}(\Lambda^{0,1}_j \otimes_{(j,J)} F^*T\overline{X})$. Since $D_F : T_F\mathcal{B}^{k,p}_{\delta_Z} \to L^p_{\delta_Z}(\Lambda^{0,1}_j \otimes_{(j,J)} F^*T\overline{X})$ is Fredholm (see [41], section 4.5), the image of $W^{k,p}_{\delta_Z}(F^*T\overline{X}) \oplus V_Z \oplus 0$ under D_F is closed and it suffices to show it is dense.

If it were not dense then by the Hahn-Banach theorem there would be a nontrivial $\beta \in L^q(\Lambda^{0,1}_j \otimes_{(j,J)} F^*T\overline{X})$ with $q^{-1} + p^{-1} = 1$, annihilating the image of D_F. This means that

$$\int_{S\setminus Z} \langle \beta, D_F\xi \rangle \, d\mathrm{vol} = 0 \tag{A.3}$$

$$\int_{S\setminus Z} \langle \beta, Y \circ dF \circ j \rangle \, d\mathrm{vol} = 0 \tag{A.4}$$

It is possible to show as in the proof of proposition 5.2.23 that unless $\beta = 0$ at an injective point of the curve one can derive a contradiction by considering some Y supported in $B_\epsilon(F(w))$ making the integral (A.4) positive. Hence β vanishes on any injective point. There is an open subset of injective points, hence the following lemma shows that β vanishes identically:

Lemma A.2.4 ([28], lemma 3.4.7)**.** *Let $p > 2$, $J \in \mathcal{J}^\ell$ and $F : S \setminus Z \to \overline{X}$ be a simple punctured finite-energy J-holomorphic curve. If there is a $\beta \in L^q_{\delta_Z}(\Lambda^{0,1}_j \otimes_{(j,J)} F^*T\overline{X})$ with $q^{-1} + p^{-1} = 1$ satifying*

$$\pi_X\xi = 0 \Rightarrow \int_{S\setminus Z} \langle \beta, D_F\xi \rangle \, dvol = 0$$

for every $\xi \in T_F\mathcal{B}^{k,p}_{\delta_Z}$ then $\beta \in W^{\ell,p}_{loc}(\Lambda^{0,1}_j \otimes_{(j,J)} F^*T\overline{X})$ and $D_F^*\xi = 0$ on $S \setminus Z$ where D_F^* is the formal adjoint of D_F. Moreover $\beta \equiv 0$ if it vanishes on a nonempty open set.

Proof. The proof from lemma 3.4.7. of [28] carries through because it is only a local regularity result. □

This finishes the proof of lemma A.2.2. □

Proof of proposition 5.2.24. The tangent space at (J, F) to $\mathcal{M}^*(A, S, Z, \rho_Z, \mathcal{J}^\ell)$ is the subspace of $(\xi, Y) \in \left(W^{\ell,p}_{\delta_Z}(F^*T\overline{X}) \oplus V_Z \oplus X_Z\right) \times \left(\mathcal{C}^\ell(\mathrm{End}(T\overline{X}, J, \eta, \omega))\right)$ satisfying $D_F\xi + Y \circ dF \circ j = 0$. The derivative of the evaluation map $\mathrm{ev}_R : \mathcal{M}^*(A, S, Z, \rho_Z, \mathcal{J}) \to R_Z$ is the projection to X_Z which is naturally identified with the tangent space at $\prod_{z\in Z} \gamma_z$ to R_Z.

For any tangent vector $v_Z \in T_{\mathrm{ev}_R(J,F)} R_Z$, lemma A.2.2 applied to each component $S_j \setminus Z_j$ of $S \setminus Z$ gives us a tangent field (ξ_j, Y_j) to the universal moduli space of that subcurve such that $\sum_j \xi_j$ projects to v_Z in X_Z. Choose the points $w_j \in S_j \setminus Z_j$ such that the balls $B_\epsilon(F(w_j))$ are pairwise disjoint and do not intersect $F(\Sigma_k)$ unless $k = j$. This choice is possible by simplicity of F.

Finally, let ξ be in $W^{\ell,p}_{\delta_Z}(F^*T\overline{X}) \oplus V_Z \oplus X_Z$ such that $\xi|_{S_j \setminus Z_j} = \xi_j$ and $Y = \sum_j Y_j$. The pair (ξ, Y) is then in the preimage $d\mathrm{ev}_R^{-1}(v_Z)$ and $d\mathrm{ev}_R$ is surjective. □

Proof of theorems 11, 12. These theorems are standard applications of the Sard-Smale theorem given the above propositions. For full details of these arguments, compare with [28], theorem 3.1.5.(II). The formula for the dimension of the smooth moduli space comes from a Fredholm index formula for D_F, see [6], section 5. □

Appendix B

Gromov's theory of pseudoholomorphic curves

B.1 Basics

Definition B.1.1. *Let (X, J) be an almost complex manifold. A J-holomorphic curve in X is a smooth map $u : \Sigma \to X$ from a Riemann surface (Σ, j) into X whose derivative Du satisfies*

$$\overline{\partial}_J u := \frac{1}{2}\left(Du + J \circ Du \circ j\right) = 0$$

u is called multiply-covered *if it factors through a branched cover of Riemann surfaces and* simple *otherwise. We concentrate on the case where Σ has genus 0.*

We think of $\overline{\partial}_J$ as a section of the infinite-dimensional vector bundle

$$\begin{array}{ccc} \mathcal{E} & \longleftarrow & \mathcal{E}_u = \Omega^{0,1}_J(u^*TX) \\ \big\downarrow & & \\ \mathcal{C}^\infty(S^2, X; A) & & \end{array}$$

where $\mathcal{C}^\infty(S^2, X; A)$ is the space of smooth maps u from the sphere into X for which $u_*[S^2] = A \in H_2(X; \mathbb{Z})$.

Definition B.1.2 (Moduli spaces of J-holomorphic spheres)**.** *We write*

- *$\widetilde{\mathcal{M}}(A, J)$ for the space $\overline{\partial}_J^{-1}(0)$.*
- *$\mathcal{M}(A, J)$ for the quotient of $\widetilde{\mathcal{M}}(A, J)$ by the reparametrisation action of $PSL(2, \mathbb{C})$ on genus 0 curves. This is called the moduli space of (unparametrised) J-holomorphic spheres in the class A.*

- $\widetilde{\mathcal{M}}^*(A, J)$ *for the space of simple curves in* $\widetilde{\mathcal{M}}(A, J)$*.*

- $\mathcal{M}^*(A, J)$ *for the quotient of this space by reparametrisations. This is called the moduli space of (unparametrised) simple J-holomorphic spheres in the class A.*

Definition B.1.3 (Regular almost complex structures)**.** *Let $d_u\overline{\partial}_J$ denote the linearisation of $\overline{\partial}_J$ at a point $u \in \mathcal{C}^\infty(S^2, X; A)$. If u is a J-holomorphic curve then $\overline{\partial}_J(u) = 0$ so that $T_{(u,0)}\mathcal{E}$ can be naturally identified with $\mathcal{E}_u \oplus T_u\mathcal{C}^\infty(S^2, X; A)$. Write $pr_{\mathcal{E}_u}$ for the projection to the subspace $\mathcal{E}_u$. A J-holomorphic curve u is said to be* regular *if $D_u\overline{\partial}_J = pr_{\mathcal{E}_u} \circ d_u\overline{\partial}_J$ is surjective as a map between suitable Sobolev completions of $T_u\mathcal{C}^\infty(S^2, X; A)$ and $\mathcal{E}_u$ and J is called a* regular almost complex structure *for the homology class A if any J-holomorphic curve $u \in \widetilde{\mathcal{M}}^*(A, J)$ is regular.*

Theorem 15 ([28], theorem 3.1.5)**.** *For a regular almost complex structure J, the space $\mathcal{M}^*(A, J)$ is a manifold of dimension $2n + 2\langle c_1(X), A\rangle$, equipped with a natural smooth structure.*

In fact, we will mostly work with moduli spaces of *stable maps*. These have the disadvantage that they do not possess a canonical manifold structure (smooth or topological), and are just Hausdorff topological spaces. However, they compensate by having good compactness properties. We will henceforth require that J is compatible with a given symplectic form ω on X, that is $\omega(JX, JY) = \omega(X, Y)$ for all X, Y and $\omega(X, JX) > 0$ for $X \neq 0$. In this setting, the moduli spaces of stable maps will be compact. Denote the space of ω-compatible J by $\mathcal{J}$.

Recall that a tree is a set T of vertices connected by edges E to form a graph with no cycles. A k-labelling of T is an assignment of the integers $\{1, \ldots, k\}$ to the vertices of T, written $i \mapsto \alpha_i \in T$. We write $\mathrm{dom}(u)$ for the domain of a map u.

Definition B.1.4. *A genus 0 J-holomorphic stable map with k marked points modelled on a k-labelled tree T is a collection of J-holomorphic spheres $u_\alpha : S^2 \to X$, one for each vertex $\alpha \in T$, with:*

- *marked points $z_i \in dom(u_{\alpha_i})$ for $i = 1, \ldots, k$*

- *nodal points $z_{\alpha\beta} \in dom(u_\alpha)$ for each oriented edge $\alpha\beta \in E$ such that $u_\alpha(z_{\alpha\beta}) = u_\beta(z_{\beta\alpha})$.*

We require all marked and nodal points to be distinct. The final requirement on this data is stability*: that the number of* special points *(i.e. marked or nodal points) on $dom(u_\alpha)$ is at least 3 if u_α is a constant map. Such a constant component is called a* ghost bubble.

The stable map $(\mathbf{u}, \mathbf{z})$ *is said to represent a homology class* $A \in H_2(X; \mathbb{Z})$ *if*

$$A = \sum_{\alpha \in T} (u_\alpha)_* [S^2].$$

To define a moduli space of stable maps we need a suitable notion of equivalence up to reparametrisation. We say $(\mathbf{u}, \mathbf{z})$ and $(\mathbf{u}', \mathbf{z}')$ (modelled on labelled trees T and T' respectively) are equivalent if there is an isomorphism $f : T \to T'$ of trees and a reparametrisation $\phi_\alpha \in PSL(2, \mathbb{C})$ for each $\alpha \in T$ for which

$$u'_{f(\alpha)} \circ \phi_\alpha = u_\alpha, \;\; z'_{f(\alpha)f(\beta)} = \phi_\alpha(z_{\alpha\beta}), \;\; z'_i = \phi_{\alpha_i}(z_i)$$

For each k-labelled tree T this gives a reparametrisation group G_T for the space of stable maps modelled on T consisting of equivalences over automorphisms $f : T \to T$ for which $\alpha_i = f(\alpha_i)$ for all $i \in \{1, \ldots, k\}$.

Definition B.1.5. *We define*

- $\overline{\mathcal{M}}_{0,k}(X, A, J)$*, the space of equivalence classes of genus 0 J-holomorphic stable maps with k marked points representing the class A. We write* $[\mathbf{u}, \mathbf{z}]$ *for the equivalence class of the stable map* $(\mathbf{u}, \mathbf{z})$*.*
- $\widetilde{\mathcal{M}}_{0,T}(X, A, J)$*, the space of genus 0 J-holomorphic stable maps modelled on a labelled tree T.*
- $\mathcal{M}_{0,T}(X, A, J) = \widetilde{\mathcal{M}}_{0,T}(X, A, J)/G_T$*, the space of equivalence classes of genus 0 J-holomorphic stable maps modelled on a labelled tree T.*
- $\widetilde{\mathcal{M}}_{0,k}(X, A, J)$ *respectively* $\mathcal{M}_{0,k}(X, A, J)$ *to be the moduli space* $\widetilde{\mathcal{M}}_{0,T}(X, A, J)$ *respectively* $\mathcal{M}_{0,T}(X, A, J)$ *when T is the k-labelled tree with one vertex.*

We also introduce a notion of simplicity for stable maps, and suffix the notation for moduli spaces by * to denote restriction to simple stable maps.

Definition B.1.6. *A stable map* $(\mathbf{u}, \mathbf{z})$ *modelled on a tree T is simple if every non-constant component is a simple J-holomorphic map and no two distinct vertices of T give rise to non-constant maps with the same image.*

We need to equip our moduli spaces with a topology. The *Gromov topology*, defined in [28], section 5.6, will be the one relevant for Gromov-Witten theory. We recount those properties of the resulting moduli spaces which will be relevant for this paper:

Theorem 16 ([28], theorem 5.6.6). *Write $\overline{\mathcal{M}}_{0,k}(X,A,J)$ for the moduli space of genus 0 J-holomorphic stable curves with k marked points equipped with the Gromov topology. Then $\overline{\mathcal{M}}_{0,k}(X,A,J)$ is:*

- *a separable Hausdorff space,*
- *(Gromov's compactness theorem) compact,*

and the evaluation map

$$ev : \overline{\mathcal{M}}_{0,k}(X,A,J) \to X^k, \;\; ev[\mathbf{u},\mathbf{z}] = (u_{\alpha_1}(z_1), \ldots, u_{\alpha_k}(z_k))$$

and the projection which forgets the k-th marked point

$$\overline{\mathcal{M}}_{0,k}(X,A,J) \to \overline{\mathcal{M}}_{0,k-1}(X,A,J)$$

are continuous maps.

We note that

$$\overline{\mathcal{M}}_{0,k}(X,A,J) = \bigcup_T \mathcal{M}_{0,T}(X,A,J)$$

and

$$\begin{aligned} \mathcal{M}_{0,0}(X,A,J) &= \mathcal{M}(X,A,J) \\ \mathcal{M}_{0,1}(X,A,J) &= \mathcal{M}(X,A,J) \times_{PSL(2,\mathbb{C})} S^2 \end{aligned}$$

for $A \neq 0$. These equations also hold for the $\widetilde{\mathcal{M}}$ and $\mathcal{M}^*$ versions of the moduli spaces.

B.2 Pseudocycles and transversality

Definition B.2.1 (Pseudocycles). *A d-dimensional pseudocycle in a manifold X is a smooth map $f : V \to X$ from a smooth oriented d-manifold V into X such that:*

- *the image $f(V)$ has compact closure,*
- *the* limit set
$$\Omega_f := \bigcap_{\substack{K \subset V, \\ K \text{ compact}}} \overline{f(V \setminus K)}$$
is of dimension at most $\dim(V) - 2$.

Here a subset is of dimension at most k *if it is contained in the image of a map* $f : A \to X$ *where* A *is a manifold of dimension* k*.*

Definition B.2.2 (Bordism of pseudocycles)**.** *Two pseudocycles* $f_1 : V_1 \to X$ *and* $f_2 : V_2 \to X$ *are* bordant *if there is a smooth* $d+1$*-manifold* W *with boundary* $\partial W = -V_1 \cup V_2$ *and a map* $g : W \to X$ *extending* f_0 *and* f_1 *such that* Ω_g *has dimension 2 less than* W*. Such data is called a* bordism of pseudocycles*.*

To state the next lemma which concerns intersections of pseudocycles we need a notion of transversality.

Definition B.2.3 (Strong transversality)**.** *Two pseudocycles* $f_1 : V_1 \to X$ *and* $f_2 : V_2 \to X$ *are* strongly transverse *if their images are transverse wherever they intersect and neither limit set intersects the closure of the other pseudocycle, i.e.*

$$\Omega_{f_i} \cap \overline{f_j(V_j)} = \emptyset$$

Lemma B.2.4 ([28], lemma 6.5.5)**.** *Let* $f_1 : V_1 \to X$ *and* $f_2 : V_2 \to X$ *be pseudocycles of complementary dimension.*

- *There is a Baire set of diffeomorphisms* ϕ *of* X *for which* $\phi \circ f_1$ *and* f_2 *are strongly transverse.*
- *If they are strongly transverse then their intersection is finite. Define* $f_1 \cdot f_2$ *to be the sum of the local intersection numbers of* $f_1(V_1)$ *and* $f_2(V_2)$ *at their intersection points.*
- *The intersection number* $f_1 \cdot f_2$ *is independent of the pseudocycles up to bordism.*

The pseudocycles we will consider are the *Gromov-Witten pseudocycles* from the evaluation maps

$$\mathrm{ev} : \mathcal{M}^*_{0,k}(X, A, J) \to X^k$$

for certain homology classes in a symplectic Del Pezzo surface $X = \mathbb{D}_n$. In fact, all the classes considered will contain only simple curves, so the superscript * is superfluous. However, for general Gromov-Witten theory it is essential, so we leave it in here. The crucial theorem regarding these pseudocycles requires a finer notion of regularity for almost complex structures than those we have considered so far.

Definition B.2.5 (GW-regularity)**.** *Let* T *be a* k*-labelled tree with a set of oriented edges* E *and* A_α *a homology class in* $H_2(X, \mathbb{Z})$ *for each vertex* $\alpha \in T$*. Write* $A = \sum_{\alpha \in T} A_\alpha$*. Set*

$$Z(T) \subset (S^2)^{\# E} \times (S^2)^k$$

to be the set of all tuples $(z_{\alpha\beta}, z_i)$ *such that for every vertex* α*, the points* $z_{\alpha\beta}$ *and* z_i *are distinct for all* β *and* i *such that* $\alpha_i = \alpha$*.*

Let J *be an almost complex structure on* X *and consider the moduli space*

$$\mathcal{M}^*(\{A_\alpha\}, T) := \prod_{\alpha \in T} \mathcal{M}^*(A_\alpha, J)$$

J *is* regular for T and $\{A_\alpha\}$ *if*

- *each component* u_α *is a regular* J*-curve for every* $\mathbf{u} \in \mathcal{M}^*(\{A_\alpha\}, T)$*,*
- *(*edge transversality*) the map*

$$ev^E : \mathcal{M}^*(\{A_\alpha\}, T) \times Z(T) \to X^{\#E}$$

(sending $(\mathbf{u}, \mathbf{z})$ *to* $u_\alpha(z_{\alpha\beta})$ *for every oriented edge* $\alpha\beta$*) is transverse to the diagonal*

$$\Delta^E := \left\{ z_{\alpha\beta} \in X^{\#E} : z_{\alpha\beta} = z_{\beta\alpha} \right\}$$

We say J *is* GW-regular *for the class* A *if it is regular for all* T *and* $\{A_\alpha\}$ *such that* $\sum_{\alpha \in T} A_\alpha = A$*.*

Theorem 17 ([28], theorem 6.6.1)**.** *Let* (X, ω) *be a closed monotone symplectic manifold and* $0 \neq A \in H_2(X, \mathbb{Z})$*. If* J *is GW-regular then the Gromov-Witten evaluation map*

$$ev : \mathcal{M}^*_{0,k}(X, A, J) \to X^k$$

is a pseudocycle of dimension $2\dim(X) + 2\langle c_1(X), A\rangle + 2k - 6$ *whose bordism class is independent of* J *(amongst those which are GW-regular).*

Finally, we state a useful geometric criterion for GW-regularity.

Lemma B.2.6 ([28], lemma 6.2.2)**.** *The edge evaluation map* ev^E *is transverse to* Δ^E *if and only if for every simple stable map* $(\mathbf{u}, \mathbf{z}) \in \widetilde{\mathcal{M}}^*_{0,T}(\{A_\alpha\}, J)$*, every edge* $\alpha\beta \in E$ *and every pair* $v_{\alpha\beta} + v_{\beta\alpha} = 0$ *with* $v_{\alpha\beta} \in T_{u_\alpha(z_{\alpha\beta})}X$ *there are vectors*

$$\xi_\alpha \in \ker D_{u_\alpha}\overline{\partial}_J, \;\; \zeta_{\alpha\beta} \in T_{z_{\alpha\beta}}S^2$$

such that

$$v_{\alpha\beta} = \xi_\alpha(z_{\alpha\beta}) - \xi_\beta(z_{\alpha\beta}) + du_\alpha(z_{\alpha\beta})\zeta_{\alpha\beta} - du_\beta(z_{\beta\alpha})\zeta_{\beta\alpha}$$

This seemingly complicated lemma is actually a simple consequence of the formula

$$d\mathrm{ev}_{\alpha\beta}(\mathbf{u},\mathbf{z})(\xi,\zeta) = \xi_\alpha(z_{\alpha\beta}) + du_\alpha(z_{\alpha\beta})\zeta_{\alpha\beta}$$

for the derivative of the map $\mathrm{ev}_{\alpha\beta}(\mathbf{u},\mathbf{z}) = u_\alpha(z_{\alpha\beta})$.

B.3 Geometry of pseudoholomorphic curves in 4D

All the special features of pseudoholomorphic curves in dimension 4 stem from the following theorem of McDuff on their intersection properties. Here (X, J) is an arbitrary almost complex 4-manifold.

Theorem 18 (Positivity of intersections [28], theorem 2.6.3)**.** *Let u_0, u_1 be simple J-holomorphic curves in X representing homology classes A_0 and A_1. If u_0 and u_1 have geometrically distinct images on every pair of open subsets of the domains then every intersection of the images contributes a positive integer to the homological intersection number $A_0 \cdot A_1$. This number is 1 if and only if the intersections are transverse.*

Theorem 19 (Adjunction inequality [28], theorem 2.6.4)**.** *Let $u : \Sigma \to X$ be a simple J-holomorphic curve in X and $A = u_*[S^2] \in H_2(X, \mathbb{Z})$. Define*

$$\delta(u) := \#\{(a, b) \in \Sigma \times \Sigma : a \neq b \text{ and } u(a) = u(b)\}$$

Then

$$\delta(u) \leq A \cdot A - \langle c_1(X), A\rangle + \chi(\Sigma)$$

Theorem 20 (Automatic transversality, [22] or [28] lemma 3.3.3)**.** *Let $u : S^2 \to X$ be an embedded J-holomorphic sphere in an almost complex 4-manifold X. If $c_1(X)$ evaluates positively on u then u is regular.*

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