Topology and Groups

Week 8, Monday

1 Preparation

- 7.05 (Fundamental group of the circle),
- 7.06 (Group actions and covering spaces, 1),
- 7.07 (Group actions and covering spaces, 2).
- 8.01 (Lifting criterion, first 4 minutes).

2 Discussion

1. If X is simply-connected and admits a properly discontinuous G-action then we showed that $\pi_1(X/G) \cong G$. What was the map $F\colon G \to \pi_1(X/G)$ we used and why was it well-defined?
2. (PCQ) I claim that any covering space of a CW complex is itself a CW complex. How would I construct the cells of this covering space?

3 Classwork

3.1 Covers of surfaces

1. Let X be a surface of genus $g \geq 2$. For each $d \geq 2$, find a covering space $p\colon Y \to X$ such that the index of $p_*\pi_1(Y, y) \subset \pi_1(X, x)$ is equal to d. (Hint: Find a surface of even higher genus with an action of $\mathbf{Z}/d$ whose quotient is X).
2. If $p\colon Y \to X$ is a d-fold covering space of X, show that $\chi(Y) = d\chi(X)$ (where $\chi(M)$ denotes the Euler characteristic, which is the alternating

sum $a_0(M) - a_1(M) + a_2(M) + \cdots$ where $a_k(M)$ is the number of k-cells in a CW structure on M).

3. Find the Euler characteristic of a closed surface of genus g. Hence, give a necessary condition for the existence of a covering map $p\colon \Sigma_g \to \Sigma_h$ (where Σ_n denotes a closed surface of genus n). Is your condition sufficient?

3.2 Finite quotients of $SU(2)$

1. Let Γ be a finite group and suppose that Γ acts freely (i.e. $gx = x$ implies $g = 1$) by isometries on a metric space. Prove that the action is properly discontinuous.

2. Let $G = SU(2) = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} : |a|^2 + |b|^2 = 1 \right\}$. Topologically, G is homeomorphic to the 3-sphere. Show that the action of G on itself by $\rho(g)h = gh$ is free and by isometries (with respect to the subspace metric, thinking of $SU(2) \subset \mathbf{C}^2$ and giving $\mathbf{C}^2$ the Euclidean metric).

3. Together, Q2 and 3 imply that $SU(2) \to SU(2)/\Gamma$ is a covering space for any finite subgroup $\Gamma \subset SU(2)$. What does this have to do with Assessed Project 1?