Topology and Groups

Week 6, Monday

1 Preparation

- 6.01 (Braid group),
- 6.02 (Artin action).

2 Discussion

1. (PCQ) Why do braids form a group under stacking?
2. (PCQ) What is the Artin action of σ_1^{-1}?
3. Recall that each braid $(F_i(t), t)$ satisfies $F_i(0) = z_i$ and $F_i(1) = z_{s(i)}$ for some permutation $s\colon \{1, \ldots, n\} \to \{1, \ldots, n\}$. The assignment of the permutation s to the braid $(F_i(t), t)$ gives a homomorphism from the braid group B_n on n-strands to the permutation group S_n. The kernel of this homomorphism is called the *pure braid group* PB_n. Can you give me a space X such that $\pi_1(X) = PB_n$?

3 Classwork

1. Give a picture-proof that the braid relations hold:

$$\sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| > 1$$
$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}.$$

2. Calculate the Artin action of σ_1^n on $\mathbf{Z} \star \mathbf{Z} = \langle \alpha, \beta \rangle$.
3. Calculate the Artin action of $\sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_1$ on $\langle \alpha, \beta, \gamma \rangle$.

4 Surgery on 3-manifolds

Let $K \subset S^3$ be a knot. Thicken the knot slightly; you get a knotted solid torus N called a *tubular neighbourhood* of the knot. Let $C = S^3 \setminus N$ be the complement of this solid torus. Both the boundary of N and the boundary of C are homeomorphic to T^2. Let $\phi\colon \partial N \to \partial C$ be a homeomorphism and consider the space

$$S^3_{K,\phi} := (C \coprod N)/\sim, \qquad \partial N \ni x \sim \phi(x) \in \partial C.$$

In other words, $S^3_{K,\phi}$ is obtained from S^3 by cutting out a solid torus and gluing it back in with a twist, ϕ. This procedure for constructing new spaces is called *Dehn surgery*; any 3-dimensional manifold can be obtained by a sequence of Dehn surgeries on knots starting from S^3.

1. Explain how you would compute the fundamental group of $S^3_{K,\phi}$ if you knew the fundamental group of $S^3 \setminus K$. Exactly what information do you need to know about ϕ?
2. Given that $\pi_1(S^3 \setminus U) \cong \mathbf{Z}$, where U is the unknot, compute $\pi_1(S^1_{U,\phi})$ for the homeomorphism

$$\phi \begin{pmatrix} e^{i\alpha} \\ e^{i\beta} \end{pmatrix} = \begin{pmatrix} e^{ia\alpha+b\beta} \\ e^{ic\alpha+d\beta} \end{pmatrix}.$$