Topology and Groups

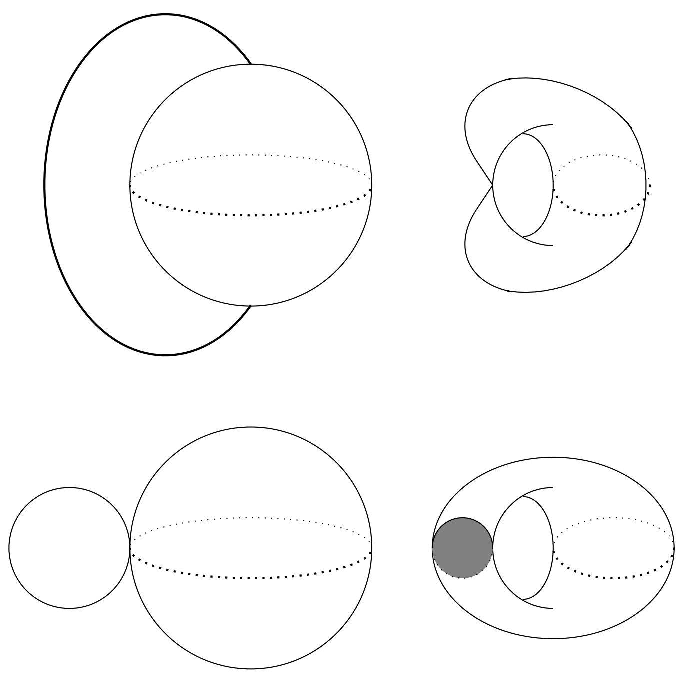
Week 5, Monday

1 Preparation

- 3.02 (Quotient topology: continuous maps),
- 4.02 (Homotopy extension property (HEP)),
- 4.03 (CW complexes have the HEP).

2 Discussion

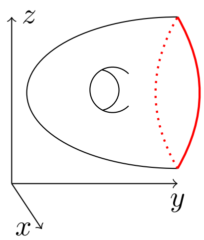
1. (PCQ) Which of the following spaces are homotopy equivalent to one another?



2. (PCQ) In the proof that a connected 1-dimensional CW complex is homotopy equivalent to a wedge of circles, where did we use that it was connected?

3. (PCQ) Let X be the space in the figure below (thought of as sitting inside $\mathbf{R}^3$) and let A be the red subset. Which of the following functions $X \to \mathbf{R}$ descends to the quotient X/A?

   - the projection to the z-axis,
   - the projection to the x-axis,
   - the projection to the y-axis?



3 Classwork

1. Show that any connected CW complex is homotopy equivalent to a CW complex with a single 0-cell.

2. Given a space A, the *cone* CA on A is the space $(A \times [0,1])/(A \times \{1\})$. Show that CA is contractible.

3. Let X be a space and let $f, g\colon S^{n-1} \to X$ be two continuous maps. Let $X(f) = X \cup_f D^n$ and $X(g) = X \cup_g D^n$ be the spaces obtained by attaching cells along f and g respectively.

   - In the case that $X = \mathbf{R}^2$, f, g are the inclusions of circles of different radii sketch $X(f)$ and $X(g)$.

   Given a homotopy H from f to g, define $X(H) = X \cup_H (D^n \times [0,1])$, in other words $(X \coprod (D^n \times [0,1]))/\sim$ where $S^{n-1} \times [0,1] \ni (p,t) \sim H(p,t)$.

   - If H is the obvious radial homotopy between f and g in the previous example, sketch $X(H)$.

- In general, motivated by your picture, how might you prove that $X(g) \simeq X(H) \simeq X(f)$?

The moral of this question is that the homotopy type of a CW complex only depends on the *homotopy classes* of the attaching maps.

4. Let X be a CW complex and $A \subset X$ be a subcomplex with inclusion map $i\colon A \to X$. Let $X \cup_i CA$ be the map which attaches CA to X by identifying a point $(a, 0) \in CA$ with the corresponding point $i(a) \in X$. Show that $X/A \simeq X \cup_i CA$.

5. Suppose that $A \subset X$ is a subcomplex such that the inclusion map $i\colon A \to X$ is nullhomotopic. Show that $X/A \simeq X \vee SA$, where SA is the *suspension* of A, defined to be $SA = (A \times [0, 1])/(A \times \{0, 1\})$.

6. If S^1 is the unknot in S^3 then what is $\pi_1(S^3/S^1)$?

4 Questionnaire

1. What do you like about this class?

2. What do you dislike about this class?

3. If you were teaching this class, what would you do?

4. If you could change one thing about this class, what would it be?

5. So far, has the amount of video you needed to watch each week been reasonable?

6. Any other comments?