Topology and Groups

Week 4, Monday

1 Preparation

- 2.04 (Connectedness, path-connectedness),
- 5.01 (Van Kampen's theorem),
- 5.02 (fundamental group of a CW complex).

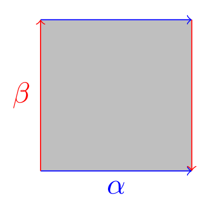
2 Connectedness, path-connectedness

True or false?

1. Every indiscrete space is connected.
2. Every indiscrete space is path-connected.
3. A subspace of a connected space is connected.

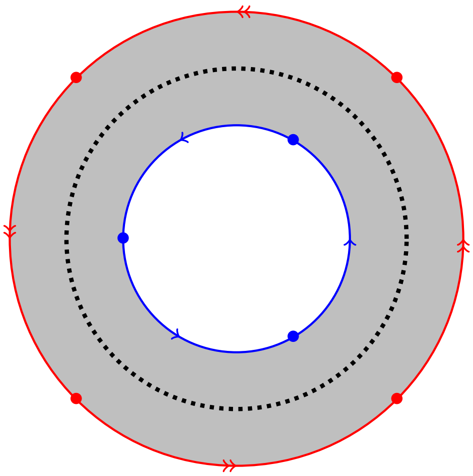
3 Van Kampen's theorem: discussion

1. (PCQ) What is a presentation for π_1 of the Klein bottle?



2. (PCQ) What about a genus 2 surface?

3. Consider the following quotient space of the annulus:



where the outer circle is divided into 4 segments and the inner circle into 3 segments. Van Kampen's theorem tells us the fundamental group (e.g. if we cut it in two along the dotted circle). What is the answer?

   (a) $\mathbf{Z}$?

   (b) Trivial?

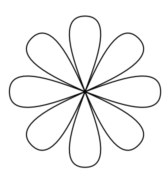
   (c) $\langle a, b \mid a^4 = b^3 \rangle$?

   (d) $\langle a, b \mid a^4 = 1,\ b^3 = 1 \rangle$?

   (e) $(\mathbf{Z}/3) \star (\mathbf{Z}/4)$?

4. Given two spaces X and Y and points $x \in X$ and $y \in Y$, the *wedge* $X \vee Y$ is defined to be the quotient space $(X \coprod Y)/\{x, y\}$. What is $\pi_1(X \vee Y, b)$, where b is the equivalence class $\{x, y\}$?

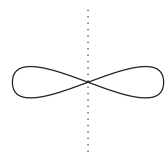
5. What is the fundamental group of a wedge of n circles? (Drawn in the case $n = 8$ below).



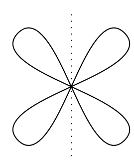
4 Classwork

The following questions will be useful when we look at knots and links (you might like to think about why).

1. Let P be a pinched torus. Write down a cell structure on P and hence (or otherwise) compute its fundamental group.
2. Consider the figure 8,

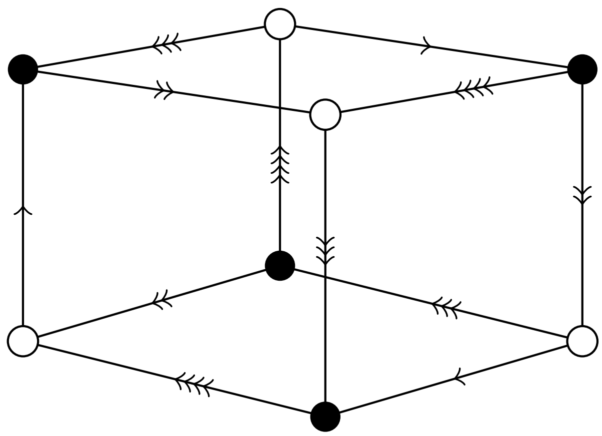
   

   and form the surface of revolution of this curve around the dotted axis. Why is this homeomorphic to the pinched torus? What is its fundamental group?
3. Find the fundamental group of the surface of revolution obtained from the following curve around the dotted axis:

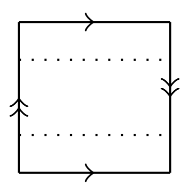
5 Van Kampen's theorem: further discussion

Take a cube and identify opposite faces with a 90 degree twist as shown below. The quotient space has a CW structure: how many cells of each dimension? How would we go about figuring out its fundamental group?



6 Classwork

1. The Klein bottle can be cut along a circle so that it falls apart into two Möbius strips:

   Use this description to give a presentation of π_1 of the Klein bottle. Explain how this presentation is related to the presentation in the presentation we found before (hint: pay attention to basepoints).

7 Assessed project 1 (due in two weeks, worth 5 marks)

What is the fundamental group of the space obtained from a solid dodecahedron by gluing its opposite faces as indicated?