SOME SIMPLE SPECTRAL SEQUENCES: GUIDED EXERCISES

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Abstract. These exercises attempt to communicate the germ of the idea of spectral sequences. You should think of them as a precursor to actually learning about spectral sequences in a more respectable place like Bott and Tu ("Differential forms in algebraic topology"), McCleary ("A user's guide to spectral sequences") or Hatcher ("Spectral sequences in algebraic topology").

The focus here is only on the formalism and basic ideas: there are no examples. The aforementioned references have lots of nice examples.

1. Two term double complex

Consider a pair of chain complexes (V, ∂_V) and (W, ∂_W) and a map $f\colon V \to W$. Define the (block lower-triangular) operator

$$D\begin{pmatrix} a \\ b \end{pmatrix} := \begin{pmatrix} \partial_V & 0 \\ f & \partial_W \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \partial_V a \\ \partial_W b + fa \end{pmatrix}$$

Question 1.

(a) Show that D is a differential (i.e. $D^2 = 0$) if and only if $f\partial_V = -\partial_W f$.

(b) Show that this, in turn, implies that f induces a map $f_*\colon H(V, \partial_V) \to H(W, \partial_W)$ on homology.

Lemma 1. *The homology of $(V \oplus W, D)$ is isomorphic to*

$$\ker(f_*) \oplus (H(W, \partial_W)/\mathrm{im}(f_*))\,.$$

Proof. First note that $\begin{pmatrix} a \\ b \end{pmatrix}$ is D-closed if both $\partial_V a = 0$ and $\partial_W b + fa = 0$ and is D-exact if $a = \partial_V a'$ and $b = \partial_W b' + fa'$.

Question 2.

(a) Show that the map

$$\psi\colon H(V \oplus W, D) \to H(V, \partial_V), \quad \psi\left[\begin{pmatrix} a \\ b \end{pmatrix}\right] = [a]$$

is well-defined.

(b) Show that the image of ψ is equal to $\ker(f_*)$.

We therefore have $H(V \oplus W, D) \cong \ker(f_*) \oplus \ker(\psi)$. The next step is to identify $\ker(\psi)$ with $H(W, \partial_W)/\mathrm{im}(f_*)$.

Question 3.

(c) Fix $\left[\begin{pmatrix} a \\ b \end{pmatrix}\right]$ in the kernel of ψ. Then $[a] = 0$, so $a = \partial_V a'$. Show that $[b - fa']$ is well-defined as an element of $H(W, \partial_W)/\mathrm{im}(f_*)$, independently of the choice of a'.

(d) Prove that

$$\phi\colon \ker(\psi) \to H(W, \partial_W)/\mathrm{im}(f_*), \quad \phi\left[\begin{pmatrix} a \\ b \end{pmatrix}\right] = [b - fa']$$

is an isomorphism.

□

In other words, you can separate your calculation into two: first calculate the ∂_V and ∂_W homology groups, then treat f_* as a remaining differential and take the homology of the complex $f_*\colon H_*(V, \partial_V) \to H_*(W, \partial_W)$.

Remark 1. What was special about $(V \oplus W, D)$ that allowed us to compute its homology this way? It was the fact that $D = \begin{pmatrix} \partial_V & 0 \\ f & \partial_W \end{pmatrix}$ is lower-triangular. In general, when a big complex splits up into a collection of subspaces with a lower-triangular differential, we can split up the computation of the homology of the total complex into a bunch of separate calculations: roughly speaking, first take the homology for the diagonal elements of the differential, then for the part of the differential just below the diagonal, etc. A spectral sequence is a way of keeping track of this sequence of calculations. In the next section we will see how this works when there are three blocks in our lower-triangular differential.

2. Three term double complex

Take a chain complex of the form $(V \oplus W \oplus X, D)$ where

$$D = \begin{pmatrix} \partial_V & 0 & 0 \\ f & \partial_W & 0 \\ h & g & \partial_X \end{pmatrix}$$

Question 4. (a) What relations between $f, g, h, \partial_V, \partial_W, \partial_X$ are implied by $D^2 = 0$?

(b) Show that $g_* f_* = 0$ so that $H(V, \partial_V) \xrightarrow{f_*} H(W, \partial_W) \xrightarrow{g_*} H(X, \partial_X)$ is a chain complex whose homology is isomorphic to

$$\ker(f_*) \oplus \ker(g_*)/\mathrm{im}(f_*) \oplus H(X, \partial_X)/\mathrm{im}(g_*)$$

As in Question 2, there is a well-defined map

$$\psi\colon H(V \oplus W \oplus Z, D) \to H(V, \partial_V), \quad \psi\left[\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right] = [a]$$

whose image is contained in $\ker(f_*)$. However, this may fail to be surjective, because we have the additional requirement on $[a]$ that $\partial_X c + gb + ha = 0$.

Question 5. If $[a] \in \ker(f_*)$, there exists $b \in W$ such that $fa = \partial_W b$.

(a) Show that in this case $[ha+gb]$ is well-defined as an element of $H(X, \partial_X)/\mathrm{im}(g_*)$ (i.e. $ha+gb$ is ∂_X-closed and the cohomology class is independent of choices up to an element of $\mathrm{im}(g_*)$). We define

$$h_2\colon \ker(f_*) \to H(X, \partial_X)/\mathrm{im}(g_*), \quad h_2[a] = [ha + gb] \mod \mathrm{im}(g_*).$$

(b) Prove that ψ is surjective onto $\ker(h_2) \subset \ker(f_*)$.

Question 6. (a) Suppose that $\left[\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right] \in H(V \oplus W \oplus Z, D)$ is in the kernel of ψ, so that $a = \partial_V a'$. Show that $[b - fa'] \in \ker(g_*)/\mathrm{im}(f_*)$ is well-defined and gives a surjection $\phi\colon \ker(\psi) \to \ker(g_*)/\mathrm{im}(f_*)$.

(b) Finally, write down a well-defined projection

$$\ker(\phi) \to \left(H(X, \partial_X)/\mathrm{im}(g_*)\right)/\mathrm{im}(h_2)$$

and prove it is an isomorphism.

The conclusion of this is

Lemma 2.

$$H(V \oplus W \oplus X, D) \cong \ker(h_2) \oplus \ker(g_*)/\mathrm{im}(f_*) \oplus \left(H(X, \partial_X)/\mathrm{im}(g_*)\right)/\mathrm{im}(h_2).$$

This may seem useless because we can't hope to keep track of h_2 in general.

However, when the complexes (V, ∂_V) etc are graded we can sometimes extract useful information. Suppose that $\partial_V, \partial_W, \partial_X$ have degree 1, that f, g have degree 0 and that h has degree -1. Then $(V[2] \oplus W[1] \oplus X, D)$ is a graded complex where the differential has degree 1 (recall that $V[2]$ means the chain complex with $V[2]_i = V[2+i]$). The differential h_2 is now a map of degree -1. Sometimes that's a really useful observation, for example if $H(V)$, $H(W)$ and $H(X)$ are all supported in even degrees, because then h_2 is necessarily zero and you can ignore it. It happens quite often (often enough to be useful) that the differentials in a spectral sequence vanish.