Symplectic Picard-Lefschetz theory

Jonny Evans

ETH Zürich

17th April 2012

- The circle of ideas I will discuss today has its roots in the study of algebraic surfaces by Picard and Lefschetz.
- The formulation I will describe was worked out in the late nineties by Donaldson.
- It has come to inform the way many people think about symplectic geometry.
- My goal (other than explaining the theory) is to get to some interesting open questions which are combinatorial in flavour, but which would have ramifications in symplectic geometry.

Plan

- Morse theory
- Complex Morse theory
 - on Riemann surfaces (branched covers)
 - on 4-manifolds (Picard-Lefschetz theory)
- Pencils
- Relation to symplectic geometry
- Lagrangian submanifolds

Morse theory

- Functions allow us to cut up manifolds.
- In the picture below we see a height function $f: T^2 \rightarrow \mathbf{R}$.



Picture courtesy of Wikipedia.

Morse theory

- The topology of the level sets changes at critical levels.
- These contain *critical points* where all derivatives of *f* vanish.



Picture courtesy of Wikipedia.

Here's an even simpler example.



• The function we're most interested in for today's purposes is

$$f(x) = x^2$$

• This has a unique critical point at the origin.



- Of course there are other critical points a function can have, for example f(x) = x³ has an inflection point at 0...
- ...but this can be perturbed slightly and replaced by two quadratic critical points.



More generally:

Lemma (Morse lemma)

For a generic smooth function

- the critical points are isolated
- in suitable local coordinates centred at a critical point, the function looks like a nondegenerate quadratic form.

In two dimensions, the three possible quadratic critical points correspond to the possible signatures of a nondegenerate real quadratic form.



Complex Morse theory

- The idea of 'complex-valued Morse functions' came from a paper of Andreotti and Fraenkel, who used it to prove the **Lefschetz hyperplane theorem**.
- Similar ideas (in a different guise) were developed by Picard and Lefschetz in their works on the topology of algebraic surfaces.
- Thanks to insight of Arnold, Donaldson, Seidel and others, we now know that symplectic geometry is the natural setting where these ideas bear fruit.
- We will start by looking at the simplest case: branched covers

$$\Sigma \to \mathbf{C} \cup \{\infty\}$$

of the Riemann sphere by a Riemann surface Σ .

Branched covers

- Let T^2 be the 2-torus embedded in \mathbf{R}^3 and poke an axis through it.
- Consider the 180° flip around this axis.



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Branched covers

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- Consider the 180° flip around this axis.



- Define two points to be equivalent if they are related by the flip.
- A point where the axis pierces T^2 forms an equivalence class by itself.
- All other points have equivalence classes of size two.
- We'll draw the quotient space of T^2 by this equivalence relation.







• In summary, we have the quotient map $T^2 o S^2$:



- You should take my pictures with an ever-increasing dose of salt.
- Just because I draw a space with an arrow down to another space doesn't mean that the preimage of a point is above that point.
- To clarify matters, we'll work out some preimages.


































We call this phenomenon *monodromy*. This is reminiscent of analytic continuation for \sqrt{z} .

- Take graph of $z \mapsto \sqrt{z}$.
- Project onto z-plane.
- This is the inverse $z \mapsto z^2$.
- This is precisely the local model of our branched cover near the singular points.
- Again, it's quadratic!



Lemma (Complex Morse lemma)

For a generic smooth map $f:\Sigma\to\Sigma',$ where Σ and Σ' are Riemann surfaces,

- the critical points are isolated,
- f looks like

$$z\mapsto z^2$$

in suitable local coordinates centred at a critical point.

We can encode this data as follows.

- A finite set *C* of *branch points* in the sphere (the critical values).
- A finite set F (the preimage of a regular point).
- A monodromy representation $\pi_1(S^2 \setminus C) \to \operatorname{Sym}(F)$.
- This representation must take simple loops which encircle a single point in C to a transposition (ab) ∈ Sym(F).

In our example:

- C is just the four branch points.
- *F* is just two points (double branched cover).
- The monodromy representation takes a small loop centred at a branch point to the permutation which swaps the two points of *F*.

From this we can reconstruct the torus.

Further examples

Example

Consider the complex variety

$$\{x^2 + y^2 = 1 : (x, y) \in \mathbf{C}^2\} \cong \mathbf{C}^*.$$

- The projection $\pi: (x, y) \mapsto x$ is a branched double cover.
- There are two critical points (at ± 1).
- Over **R**, the real part is a circle.
- The restriction of π to the real part is the simple Morse function on the circle we began with.



- This is a very important picture to bear in mind. We'll come back to it later.
- One thing to notice is that as you move along [-1,1] ⊂ C towards either endpoint, the two points in the fibre collapse to a critical point.
- Thought of another way, there are two '1-discs': $\pi^{-1}[-1,0]$ and $\pi^{-1}[0,1]$ coming out of the critical points which glue to give a circle.

Example

Consider the complex variety

$$\{x(x-1)(x-2)+y^2=1:(x,y)\in \mathbf{C}^2\}.$$

- The projection $\pi: (x, y) \mapsto x$ is a branched double cover.
- There are three critical points (at 0, 1, 2).
- The total space is a punctured torus (affine elliptic curve).
- It becomes harder to draw.

$$\{x(x-1)(x-2) + y^2 = 1 : (x,y) \in \mathbf{C}^2\} \to \mathbf{C}, \ (x,y) \mapsto x$$



- As before you can see 1-discs emerging from the critical points and gluing up to give circles.
- Though the intervals in the base meet at the middle critical value at an angle of 180°, the circles upstairs meet at 90°.
- We saw this earlier.

- It should be clear by now that these pictures are pretty inadequate: we don't see how the boundary of the punctured torus wraps twice around the boundary of the disc (as it should).
- The problem is a lack of dimensions. If we looked at the variety $\{x(x-1)(x-2) + y^2 = 1 : (x, y) \in \mathbb{C}^2\}$ in \mathbb{C}^2 and shone a light on it (projected it to a copy of \mathbb{R}^3) it would have the following silhouette:



Picard-Lefschetz theory

- Lack of imagination nonwithstanding, we will press on into the next relevant dimension, **dimension 4**.
- That is, we are interested in holomorphic maps from a complex surface to C or C ∪ {∞}.
- The moral we carry over from Morse theory is that the local model for critical points should be a nondegenerate quadratic form.
- There is only one nondegenerate quadratic form over C.

Local model

The local model for critical points is the map

$$\pi \colon \mathbf{C}^2 \to \mathbf{C}$$
$$\pi(x, y) = x^2 + y^2$$

- The regular fibres of this map are all smooth conics like the one we met earlier.
- There is one singular fibre $x^2 + y^2 = 0$.
- Since x² + y² = (x + iy)(x iy) this is a union of two complex lines meeting transversely.
- Unforunately this is hard to draw, so we draw it like a cone (upper and lower halves being the two lines).



Monodromy

- For branched covers we saw that, for the local model $z \mapsto z^2$, the monodromy simply swaps two points in the fibre.
- We want to describe the monodromy of our new local model.
- For branched covers it was clear how to transport preimages around a lifted loop.
- In our new model there's no canonical way so we need a connection.



Connection consists of:

- A field of subspaces in *T*C². Write Π_(x,y) ⊂ *T*_(x,y)C² for the subspace at (x, y) ∈ C².
- Each subspace $\Pi_{(x,y)}$ in the field projects isomorphically onto $T_{\pi(x,y)}\mathbf{C}$.



A vector at $\pi(x, y)$ in the base lifts uniquely to a horizontal vector in $\Pi_{(x,y)}$. Flowing along horizontal vector fields is called parallel transport.

- We require that when we transport a fibre around a closed loop in the base the parallel transport diffeomorphism has compact support.
- We can also ensure that this diffeomorphism is area-preserving.

So what diffeomorphism do we get?

Theorem (Picard, Lefschetz, Seidel)

The monodromy diffeomorphism for our local model is a Dehn twist.

This is best illustrated by a picture.













- The Dehn twist is both compactly-supported and area-preserving!
- Note that although the diffeomorphisms in between seem to act "noncompactly" shifting the vertical red line near the top of each fibre the monodromy around the loop *is* compactly-supported.

The proof of the theorem is long and explicit. Let me just sketch how you might visualise it.

- Each fibre $x^2 + y^2 = c$ has a projection $(x, y) \rightarrow x$.
- For $c = e^{i\theta}$ the critical points of this projection are

$$x = \pm e^{i\theta/2}$$

Let's watch the pictures again and keep track of this projection.











Hopefully that helped!

To encode a complex Morse function we now need a little more:

- A surface F (the fibre).
- A finite set C of critical points in S^2 .
- A monodromy representation $\pi_1(S^2 \setminus C) \to MCG(F)$.

The group MCG(F) is the **mapping class group** of the surface, i.e. the component group of the group of diffeomorphisms. The monodromy representation must take simple closed loops encircling a single critical point to **right-handed Dehn twists**.



- For branched covers, each critical point had a 1-disc flowing out of it.
- Here the real part of the local model is a 2-disc flowing out of the critical point, living over the positive real axis.
- The intersection of this disc with any fibre is called the **vanishing cycle** (the red circle) because it vanishes into the critical point under parallel transport.



- When two such discs meet they may glue up to give a 2-sphere. This is called a **matching cycle**.
- In this example we have projected the **quadric surface** $x^2 + y^2 + z^2 = 1$ to the *x*-coordinate.



- We can also create chains of spheres this way.
- In this example we have projected the Milnor fibre $x(x-1)(x-2) + y^2 + z^2 = 1$ to the x-coordinate.
Lefschetz fibration

Definition

A Lefschetz fibration on a compact 4-manifold X is a smooth map $f: X \to S^2$ whose critical points are isolated and locally diffeomorphic to the quadratic model $(x, y) \mapsto x^2 + y^2$.

Remark

A Lefschetz fibration is not a fibration in the sense of topology.

A source of Lefschetz fibrations

- A natural source of Lefschetz fibrations is **pencils of curves on** algebraic surfaces.
- By pencil, I just mean a (complex) 1-parameter family.
- For a moment let's work with real pencils because they're easier to visualise.

Example

The simplest pencil of curves is the pencil of lines through a point p in the real plane \mathbb{R}^2 . To get an S¹-valued function we send each point $q \neq p$ to the corresponding real half-line \overrightarrow{pq} . Equivalently we send $\operatorname{re}^{i\theta} \in \mathbb{C}$ to θ .



- We would like to extend this to the whole of **R**², but it simply doesn't work. We say *p* is a **basepoint** of the pencil.
- The fix is to replace p by a copy of S^1 so that the map extends.
- Think of all half-lines emanating from *p*. Imagine you're really looking down on a spiral staircase centred at *p* and that the steps are the half-lines. Furthermore, imagine that when you get to the top of the stairs you reappear at the bottom.





- This process is called (oriented) blowing-up of the basepoint.
- We call the resulting manifold \mathbf{R}^2 . It's actually an annulus.
- When you do this, the map $\mathbf{R}^2 \setminus \{p\} \to S^1$ extends to $\widetilde{\mathbf{R}^2}$.

• A similar trick works over **C** and replaces a point in C^2 (or any algebraic surface) by a copy of $S^2 \cong CP^1$, the space of complex lines through a point.

Corollary

A pencil of curves with at worst nodal singularities (quadratic critical points) gives rise to a Lefschetz fibration once all the basepoints are blown up.

Generating a pencil

So the question remains: how do we generate pencils of curves with at worst nodal singularities?

- Suppose that X is a smooth projective variety. This means it's cut out of a complex projective space CP^N by a finite system of polynomials.
- Remember that CP^N is a compactification of C^N. The compactification locus is a hyperplane CP^{N-1} (just like CP¹ = C ∪ {∞}.
- Note that the hyperplanes through a fixed point p form a copy of \mathbf{CP}^{N-2} .
- A pencil of hyperplanes is just a complex line $C \subset \mathbf{CP}^{N-2}$.

Theorem (Bertini's theorem)

Fix a smooth projective variety $X \subset \mathbb{CP}^N$. Then for almost every point $p \in \mathbb{CP}^N$ and almost every pencil of hyperplanes C through p, the intersections $T \cap X$ for $T \in C$ define a pencil of curves in X with at worst nodal singularities.

This is an algebro-geometric version of the Morse lemma. It says that by perturbing the pencil of hyperplanes you can make the intersections $T \cap X$ transverse except at a finite collection of quadratic critical points.

Example (Pencil of cubics)

Let P and Q be homogeneous cubic polynomials in three variables. Then

$$\lambda P + \mu Q = 0$$

defines^a a pencil of cubic curves in \mathbb{CP}^2 as λ and μ vary (the resulting cubic depends only on the ratio $[\lambda : \mu] \in \mathbb{CP}^1$.

^aThis arises as in Bertini's theorem from a pencil of hyperplane section of the cubic Veronese embedding of CP^2 into CP^9 .

Pencil of plane cubics



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- You can see the nine basepoints where all the cubics intersect.
- You can blow these up and obtain a space $\overline{\mathbf{CP}}{}^2{}_9$ with a Lefschetz fibration.
- It turns out that there are twelve nodal cubics in the family. The Lefschetz fibration therefore has twelve critical points.

Area-preserving condition

- I want to explain now why I required the monodromy maps to preserve area.
- The more structure our monodromy maps preserve, the more useful they are (e.g. for distinguishing two fibrations).
- In a family of complex projective curves there are two natural structures on the fibres: the complex structure and the area form.
- I'll argue first that the complex structure cannot be preserved by monodromy maps.

- In the above family of cubics, not all members are isomorphic as complex curves.
- Not only are some singular, but the isomorphism type even varies over the regular fibres.





Image from C. McMullen's website



- If we have a Lefschetz fibration $X \to S^2$ whose fibres are elliptic curves then there is a map from $S^2 \to \mathcal{M}$ (taking each fibre to its isomorphism type).
- In our cubic pencil example, this map has degree twelve, explaining the twelve nodal fibres.
- So no matter how hard we try, the complex structure of the fibres is always going to vary (just for topological reasons it has to wrap twelve times around \mathcal{M}).
- However, it's easy to check that each fibre has the same total area.
- An area-preserving map is the best we can hope for.

In higher dimensions, the area form is replaced by a **symplectic form**. This is a nondegenerate closed 2-form.

"Definition"

A 2-form ω on X is something you can integrate over 2-dimensional submanifolds $\Sigma \subset X$ to get a number,

$$\int_{\Sigma} \omega$$

ω is closed if

$$\int_{\Sigma} \omega = \int_{\Sigma'} \omega$$

for any perturbation Σ' of Σ .

- You can think of it as antisymmetric quadratic form at each point. This restricts to an area form on Σ which is what you integrate.
- Nondegeneracy means nondegeneracy of this quadratic form.

Theorem (Arnold)

There is a higher-dimensional version of a Dehn twist (the monodromy of a quadratic map in higher dimensions) which preserves the symplectic form.

Note that the symplectic form on the fibres in the local model

$$(z_1, z_2, \ldots, z_n) \mapsto z_1^2 + \cdots + z_n^2$$

is the restriction to fibres of $dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n$ where $z_k = x_k + iy_k$.

- Having a monodromy which is a diffeomorphism preserving a symplectic form is much stronger than just having a monodromy which is a diffeomorphism.
- There are many examples of symplectic diffeomorphisms which are connected by a path of diffeomorphisms but not by a path of symplectic diffeomorphisms.

The relation to symplectic geometry

In our complex projective examples like the cubic pencil, the symplectic form on the fibres is inherited from an ambient symplectic form on X. In fact...

Theorem (Gompf)

Suppose X is a closed 4-manifold which admits a Lefschetz fibration $f: X \rightarrow S^2$. Suppose moreover that the fibres are homologically nontrivial. Then there is a symplectic form on X which makes the fibres into symplectic curves. This form is unique up to deformation.

i.e. to specify a symplectic manifold, all you need to do is to write down the monodromy representation of a Lefschetz fibration.

This theorem is proved by a very explicit construction. It has a remarkable converse.

Theorem (Donaldson)

Given a closed symplectic 4-manifold X there is a finite set of basepoints such that the blow-up \tilde{X} admits a Lefschetz fibration $\tilde{X} \to S^2$ with symplectic fibres.

i.e. if you write down all monodromy representations, you have all symplectic manifolds (modulo blowing up/down).

Let's write down the monodromy representation for our cubic example.



This theorem has been advertised as a reduction of the classification of symplectic 4-manifolds to algebraic problems in the mapping class groups of surfaces. This means two things:

- If you start with the data of a Lefschetz fibration (monodromy representation) then you get a symplectic manifold via Gompf.
- You get all symplectic manifolds this way via Donaldson (modulo blowing up).

It turns out to be incredibly difficult to make use of this as a classification scheme:

- there is no good way of generating monodromy representations other than taking a symplectic manifold and finding a Lefschetz fibration!
- once you have two Lefschetz fibrations, it's hard to tell when they represent the same symplectic manifold.

Indeed, it's often easier to prove theorems about mapping class groups using symplectic geometry than vice versa.

Theorem (Akbulut-Ozbgci, Stipsicz)

Consider the mapping class group of a surface with boundary. It is not possible to write the identity as a nontrivial word of right-handed Dehn twists.

One way to prove this (not the only way) is to use the putative word to construct an "impossible symplectic manifold" which contradicts what we know about the Seiberg-Witten invariants of symplectic 4-manifolds.

• See Ozbagci-Stipsicz's wonderful book "Surgery on Contact 3-Manifolds and Stein Surfaces" for more cool stuff like this. Here is a theorem about MCGs which really seems to need the Lefschetz fibration picture and which can be proved in the same way:

Theorem (Smith 1999)

There is no relation in the MCG of a closed Riemann surface which involves only right-handed Dehn twists around simple closed curves which separate the surface. I said I'd give interesting combinatorial open problems. I think the second of these points is harder to approach, so let me just say

Question

Is there a good algebraic way of coming up with monodromy representations without recourse to symplectic geometry?

A monodromy representation is a representation of $\pi_1(S^2 \setminus C)$ in a mapping class group. The group on the left is generated by simple loops $\gamma_1, \ldots, \gamma_k$ encircling the critical points modulo the relation that $\gamma_1 \cdot \gamma_2 \cdots \gamma_k = 1$. So you need to:

Question (Reformulation)

Find factorisations of the identity (in a mapping class group) into right-handed Dehn twists.

Despite these difficulties, the Lefschetz fibration picture is absolutely fundamental to the way many people think about symplectic geometry.

 It provides the possibility of proving things by "dimensional induction":

Example

For example Paul Seidel computes Fukaya categories of K3 surfaces by using a genus 3 Lefschetz fibration and reducing the computation to a genus 3 curve.

 It provides a natural setting for visualising and computing with one of the most important objects in symplectic geometry: Lagrangian submanifolds.

Lagrangian submanifolds

Remember that a symplectic form on a 2*n*-manifold X consists of an antisymmetric quadratic form ω at each point of X.

Definition

A Lagrangian submanifold $L \subset X$ is an n-dimensional submanifold whose tangent spaces are null spaces for ω , i.e.

$$\omega(X,Y)=0$$

for all tangent vectors X, Y to L. In particular $\int_L \omega = 0$.

Example

The simplest example of a Lagrangian submanifold is a circle in a Riemann surface.

We have actually already seen some examples of Lagrangian submanifolds in symplectic 4-manifolds! Remember our parallel transport maps?



The way the connection is constructed is:

- Consider the ambient symplectic form.
- Notice that each fibre F is symplectic.
- ω is a 2-form so it defines an ω -orthogonal complement $T_p F^{\omega}$, i.e.

$$T_p X = T_p F \oplus T_p F^{\omega}$$

- $T_p F^{\omega}$ consists of vectors Y for which $\omega(X, Y) = 0$ when X is vertical.
- These are the blue planes of our symplectic connection.

- In particular, the horizontal vectors are symplectically orthogonal to vertical vectors.
- That means that a 1-dimensional Lagrangian in the fibre traces out a 2-dimensional Lagrangian *L* over a path!
- Remember that to be Lagrangian you just need $\omega(X, Y) = 0$ for all X, Y tangent to L.
- As L is 2-d, it's enough to check for X vertical and Y horizontal.








Vanishing cycles are Lagrangian circles.



The 2-discs they trace out are Lagrangian!

In particular the matching cycles that form when two vanishing cycles agree are also Lagrangian.



Theorem (Auroux-Muñoz-Presas)

Any Lagrangian sphere in a symplectic 4-manifold arises this way (for some Lefschetz fibration).

A more manageable combinatorial problem would be:

Question (Donaldson-Auroux)

Given the combinatorial data of a Lefschetz fibration, write down all the matching cycles.

There might be infinitely many, but some of these might be isotopic Lagrangian spheres.

- Many people are interested in Lagrangian tori.
- Historically this is because in integrable systems, motions are constrained to live on invariant Lagrangian tori which foliate phase space.
- For instance, Yuri Chekanov and your very own Felix Schlenk have discovered a host of interesting Lagrangian tori even in the most innocuous symplectic manifolds (Cⁿ, projective spaces, products of spheres).

How might one obtain a Lagrangian torus from a Lefschetz fibration?

Idea

- Take a vanishing cycle.
- Transport it around a closed loop in the base.
- Hope that it matches itself when it comes back!

We have already seen one example where this works.



For a second let's change coordinates

$$x' = x + iy$$
$$y' = x - iy$$

so that the projection is

$$(x',y')\mapsto x'y'$$

The torus we're seeing is the **Clifford torus**:

$$\{(e^{i heta},e^{i\phi})\}\in {f C}^2$$

which projects to the unit circle.

- You could also transport the vanishing cycle around a circle which does not enclose the singular point.
- This gives **Chekanov's torus**, the very first exotic Lagrangian torus found in **C**².



One might hope to construct more tori this way.

- By a process called *stabilisation* it's relatively easy to construct Lefschetz fibrations on **C**².
- For example there is a fibration whose fibre is a 4-punctured sphere and whose vanishing cycles are the three curves in the picture.



- There is a miraculous relation in the mapping class group of a 4-punctured sphere called the **lantern relation**.
- (It arises from the monodromy of a pencil of conics in CP²!)
- From this relation it is possible to deduce that the murky yellow curve is preserved (up to isotopy) by the monodromy of this Lefschetz fibration.
- Therefore it yields a Lagrangian torus in ${\bf C}^2!$ (non-monotone, for those who care)



Question

Find more Lagrangian tori this way! Give formulae for their basic invariants (i.e. Maslov class and symplectic area class). When are they monotone?

I hope I've given you a comprehensible (though far from comprehensive) overview of the area:

- where it came from,
- some of the basic geometric ideas,
- some of the (more combinatorially flavoured) interesting open problems.

. Thank you for your attention.