Methods 3 - Question Sheet 8

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Question 1. (10 marks)
Solve the wave equation $\partial_t^2 \phi = \partial_x^2 \phi$ with the initial conditions

(a) * $\phi(x, 0) = x \qquad \partial_t \phi(x, 0) = x^2.$

(b) $\phi(x, 0) = x^2 \qquad \partial_t \phi(x, 0) = x.$

(c) $\phi(x, 0) = 0 \qquad \partial_t \phi(x, 0) = k \sin kx.$

(d) * $\phi(x, 0) = \begin{cases} 0 & \text{if } |x| > 1 \\ 1 & \text{if } |x| \leq 1, \end{cases} \qquad \partial_t \phi(x, 0) = \begin{cases} 0 & \text{if } |x| > 1 \\ -1 & \text{if } |x| \leq 1. \end{cases}$

In this last case illustrate your answer with a spacetime diagram.

Question 2. (10 marks for * parts)
For each of the following equations:

(i) * Find coordinates u, v such that the left-hand side of the equation becomes $\partial_u \partial_v F$.

(ii) * Find the general solution of the equation.

(iii) * Find the particular solution for the given initial conditions.

(a) * $F_{xx} + 5F_{xy} + 6F_{yy} = 0$, $F(x, 0) = \sin(x)$, $\partial_y F(x, 0) = \cos(x)$.

(b) * $2F_{xx} + 3F_{xy} + F_{yy} = x$, $F(x, 0) = x$, $\partial_y F(x, 0) = 0$.

(c) $F_{xy} + F_{yy} = \sin x$, $F(x, 0) = x$, $\partial_y F(x, 0) = x$.

Question 3.
Consider the parabolic differential equation $F_{xx} + 2F_{xy} + F_{yy} = 0$. Let

$$u(x, y) = Px + Qy, \qquad v(x, y) = Rx + Sy, \qquad PS - QR \neq 0$$

be a linear change of coordinates. Find a choice[1] of P, Q, R, S so that

$$F_{uu} = F_{xx} + 2F_{xy} + F_{yy} = 0$$

and hence find the general solution of the given parabolic equation in terms of x and y.

[1] There are many choices that will work! Make sure that your choice satisfies $PS - QR \neq 0$ otherwise it's not a valid (invertible) change of coordinates.

Question 4.
Find complex coordinates $(u(x,y), v(x,y))$ so that the Laplace equation $\phi_{xx} + \phi_{yy} = 0$ becomes $\phi_{uv} = 0$. Deduce that a solution of Laplace's equation can be written $\phi(x,y) = F(u) + G(v)$ for some complex functions F, G. Express the solutions $\phi(x,y) = xy$ and $\phi(x,y) = \sin nx \sinh ny$ in this form.

Question 5. *This question is a little different: in each case we study a function $y(x,t)$ where $x \geq 0$ as well as $t \geq 0$ and so we impose a boundary condition along $x = 0$ as well as the usual boundary condition along $t = 0$.*

A prisoner is attached to an infinitely long chain parametrised by a coordinate $x \in [0, \infty)$. At time $t = 0$ the prisoner starts jumping up and down on the spot $x = 0$ so that his height at time $t \geq 0$ is $y(0,t) = 1 - \cos(t)$. The chain starts off with

$$y(x,0) = 0, \; x \geq 0 \qquad \partial_t y(x,0) = 0, \; x \geq 0,$$

and obeys the wave equation

$$y_{tt} = y_{xx}.$$

By substituting D'Alembert's solution $y(x,t) = F(x+t) + G(x-t)$ into the initial conditions show that for some constant k, $F(z) = k$ and $G(z) = -k$ for all $z \geq 0$. Using the condition $y(0,t) = 1 - \cos t$ for $t \geq 0$, deduce that $G(z) = 1 - k - \cos(z)$ for $z < 0$. Find $y(x,t)$ for all $x \geq 0$, $t \geq 0$ (be careful to separate the cases $x - t \leq 0$ and $x - t \geq 0$) and sketch $y(x, 3\pi)$.

A prison guard is sitting at $x = 8$, watching the chain. At what point does he notice that the prisoner is jumping up and down?