Methods 3 - Question Sheet 7

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Question 1. (9 marks for * parts)
Give the general solutions for the following linear partial differential equations and, in each case, find the particular solution satisfying $F(s, 0) = s$.

(a) $F_x - F_y = F$,

(b) * $2F_x + 3F_y = x^2$,

(c) $F_x + 5F_y = xy$,

(d) $xF_x + F_y = 0$,

(e) * $yF_x + xF_y = F$,

(f) $e^y F_x - F_y = xF$.

Question 2. (9 marks for * parts)
For each of the following initial value problems:

(i) try to find the best collection of adjectives to describe the equation at hand (e.g. inhomogeneous linear, quasilinear, linear with constant coefficients,...);

(ii) * write down the characteristic vector field in $\mathbf{R}^3$ and find the characteristic curves passing through the initial condition;

(iii) * give the solution to this initial value problem implicitly as a solution surface $(s, t) \mapsto (x(s, t), y(s, t), z(s, t))$;

(iv) * find and sketch the caustic of the solution surface;

(v) using a computer, plot (1) the solution surface; (2) some of the projections to the xy-plane of the characteristic curves;

(vi) * find a function whose graph is equal to the solution surface (remember to specify the domain on which this function is defined).

(a) $FF_x - F_y = y$, $F(s, 0) = s^2$.

(b) $F_x - F_y = 1$, $F(s, s) = s$.

(c) * $(x + F)F_x + F_y = F$, $F(s, 0) = s$.

(d) $xyF_x - F_y + F^2 = 0$, $F(s, 0) = s$.

(e) * $xF_x - FF_y = 1$, $F(s, 0) = 0$.

(f) $x^2F - F_x - xF_y = 0$, $F(s, s) = s$.

Question 3. (2 marks)
In this question, we will prove that the Burgers equation

$$\partial_t u + u\partial_x u = 0$$

is satisfied by the velocity field of a non-viscous fluid in one dimension.

Suppose that the real line is filled with a fluid whose particles at point x are moving with velocity $u(t, x)$ at time t. Suppose that the particles don't interact with one another or experience any external force (so by Newton's law, they have zero acceleration). Let $\gamma(t)$ be the path of one of the fluid particles so that $\dot{\gamma}(t) = u(t, \gamma(t))$. Given that γ has no acceleration, deduce that u satisfies the Burgers equation.

Question 4.
For some function $G(x, y)$, consider the linear PDE

$$-y\partial_x F + x\partial_y F = G(x, y)$$

with the initial condition $F(x, 0) = 0$ for $x > 0$. Show that this initial-value problem has a single-valued solution on $\mathbf{R}^2 \setminus \{0\}$ if and only if $\int_0^{2\pi} G(A\cos\theta, A\sin\theta)d\theta = 0$ for all A. For $G(x, y) = x$ find this solution explicitly.