Methods 3 - Question Sheet 6

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In this sheet ϕ_x denotes $\dfrac{\partial \phi}{\partial x}$.

Question 1. (10 marks for * parts)
Find the Euler-Lagrange equation for the following functionals (we have only written the Lagrangian in each case - if you need to, assume that the domain of integration is $(x, y) \in [0,1]^2$)

(a) * $\frac{1}{2}\left(\phi_x^2 + \phi_y^2\right) + \frac{1}{2}K\phi^2$ (where K is a constant),

(b) * $\phi^2\phi_x^2 + \phi_y^2$,

(c) * $\phi_x\phi_y$ (in this case, also find the general solution to the Euler-Lagrange equation),

(d) $\frac{1}{2}\left(\phi_x^2 + \phi_y^2\right)$ subject to the constraint $\int \phi^2 dxdy = K$.

(e) $\frac{1}{\phi_x} + \frac{1}{\phi_y}$.

Question 2.
A string has its endpoints fixed at $(0,0)$ and $(L,0)$. If its height at x and time t is $\phi(x,t)$ then (to a good approximation) its total kinetic energy is $E(\phi) = \frac{\rho}{2}\int_0^L \phi_t^2 dx$ and its total potential energy (coming from stretching tension) is $T(\phi) = \frac{\tau}{2}\int_0^1 \phi_x^2 dx$. The string moves to minimise the integral

$$\int_0^1 (E(\phi) - T(\phi))dt.$$

Show that the string obeys the *wave equation*

$$\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = \frac{\partial^2 \phi}{\partial x^2}$$

where $c = \sqrt{\tau/\rho}$.

Question 3. (10 marks)

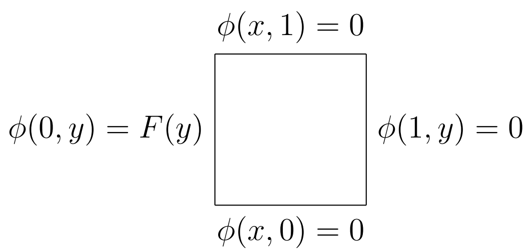
(a) Find the half-range sine series of the function

$$F(x) = \begin{cases} x^2 & \text{if } x \in [0, 1/2] \\ (x-1)^2 & \text{if } x \in [1/2, 1] \end{cases}$$

(b) Derive the Euler-Lagrange equation for the following functional

$$\int_0^1 \int_0^1 \left(\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 \right) dx dy$$

(c) Solve this Euler-Lagrange equation given the boundary values



where F is defined in part (a) of the question.

Question 4.
Show that the Euler-Lagrange equation for the functional

$$\int_0^1 \int_0^1 \sqrt{1 + \left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2} dx dy$$

is the *minimal surface equation*

$$\frac{\partial^2 \phi}{\partial x^2} \left(1 + \left(\frac{\partial \phi}{\partial y}\right)^2 \right) + \frac{\partial^2 \phi}{\partial y^2} \left(1 + \left(\frac{\partial \phi}{\partial x}\right)^2 \right) = 2 \frac{\partial^2 \phi}{\partial x \partial y} \frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial y}.$$

Question 5. Check that the function $\phi(x, y) = \sqrt{1 - x^2 - y^2}$ satisfies the *constant mean curvature equation*

$$-\lambda = \frac{\partial}{\partial x} \left(\frac{\phi_x}{\sqrt{1 + \phi_x^2 + \phi_y^2}} \right) + \frac{\partial}{\partial y} \left(\frac{\phi_y}{\sqrt{1 + \phi_x^2 + \phi_y^2}} \right)$$

for a suitable value of λ. This proves that hemispherical soap bubbles can exist!