Methods 3 - Question Sheet 5

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Question 1. (10 marks for * parts)
Solve the Euler-Lagrange equation for the following constrained functionals:

(a) * $F(y) = \int_0^{\pi/2}((y')^2 + 2xyy')dx$ subject to $\int_0^{\pi/2} ydx = K$ and the boundary conditions $y(0) = 0 = y(\pi/2)$.

(b) * $F(y) = \int_0^1((y')^2 + y^2)dx$ subject to $\int_0^1 xy = \frac{1}{6} + \frac{1}{e}$ and the boundary conditions $y(0) = 0$, $y(1) = \sinh(1) + 1/2$.

(c) $F(y) = \int_0^1 \sqrt{1 + (y')^2}dx$ subject to $\int_0^1 \sqrt{y}dx = K$ and the boundary conditions $y(0) = A$, $y(1) = B$ (you need not compute any constants in the solution and may leave the solution in implicit form).

(d) $F(y) = \int_0^{\pi/2}((y')^2 - y^2)dx$ subject to $\int_0^{\pi/2} ydx = 6 - \pi$ and the boundary conditions $y(0) = 1 = y(\pi/2)$.

(e) $F(y) = \int_a^b (y')^2 dx$ subject to $\int_a^b \frac{dx}{y} = K$ (give your solution in implicit form, specifying x in terms of y).

Question 2. (5 marks)
Suppose that $(x(t), y(t))$ is a vector-valued function of t such that $(x(a), y(a)) = (x(b), y(b)) = (0, 0)$. If $L(t, x, y, \dot{x}, \dot{y})$ is a Lagrangian and $F(x, y) = \int_a^b L(t, x, y, \dot{x}, \dot{y})dt$ is the corresponding functional, show that $(x(t), y(t))$ is a critical point of F if and only if the equations

$$\frac{\partial L}{\partial x} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}, \quad \frac{\partial L}{\partial y} = \frac{d}{dt}\frac{\partial L}{\partial \dot{y}}$$

both hold.

Question 3. (5 marks)
Let P be a polynomial of even degree 2 or more and consider the functional

$$F(y) = \int_0^1 P(y')dx$$

for functions y satisfying $y(0) = 0$, $y(1) = 1$. Find the Euler-Lagrange equation and show that if y is a solution then y' is constant. Deduce that $y(x) = x$.

Hint: When computing the Euler-Lagrange equation, use the chain rule. Your Euler-Lagrange equation should involve the derivative P' of P.

Question 4.

A smooth probability distribution on $\mathbf{R}$ with second moment σ^2 is a smooth function $\rho\colon \mathbf{R} \to [0,\infty)$ satisfying

$$(\star) \quad \int_{\mathbf{R}} \rho(x) = 1, \qquad (\star\star) \quad \int_{\mathbf{R}} x^2 \rho(x) dx = \sigma^2.$$

Show that if ρ is a smooth probability distribution maximising the *entropy functional*

$$S(\rho) = -\int_{\mathbf{R}} \rho(x) \ln(\rho(x)) dx$$

amongst all smooth probability distributions with second moment σ^2 then

$$\rho(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right).$$

Hint: Introduce two Lagrange multipliers: one for $(\star)$ *and one for* $(\star\star)$*. When imposing the constraints* $(\star)$ *and* $(\star\star)$ *it may help to remember that* $\int_{\mathbf{R}} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$ *and* $\int_{\mathbf{R}} x^2 e^{-ax^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{a^3}}$*.*