Methods 3 - Question Sheet 4

J. Evans

Question 1. (13 marks for * parts)
Solve the Euler-Lagrange equation for the following variational problems; you may use Beltrami's identity, where appropriate.

(a) * $\int_0^1 \left(x^2 y + \frac{(y')^2}{2}\right) dx$ subject to $y(0) = y(1) = 0$.

(b) * $\int_0^1 \sqrt{y(1+(y')^2)}dx$ (the general solution - leave constants undetermined).

(c) $\int_0^1 \sqrt{(1+y)(1+(y')^2)}dx$ (the general solution - leave constants undetermined).

(d) $\int_0^{\pi/4}(y')^2 \cos^2 x dx$ subject to $y(0) = 0$, $y(\pi/4) = 1$.

(e) $\int_0^{\pi} \left((y')^2 + (\cos^2 x - \sin x)y^2\right) dx$ subject to $y(0) = y(\pi) = 1$ *(Hint: Compute* $\dfrac{d^2(e^{\sin x})}{dx^2}$*.)*

Question 2. (7 marks)

Consider the functional

$$F(y) = \int_a^b \frac{\sqrt{1+(y')^2}}{y} dx$$

for functions y satisfying $y(a) = A$, $y(b) = B$. Find the general solution to the Euler-Lagrange equation for this functional and show that if y is a solution then the graph

$$\{(x, y(x)) \ : \ x \in [a, b]\}$$

is a segment of a circle

$$y^2 + (x - C)^2 = D$$

centred on the x-axis.

Remark 1. This is the equation for a *geodesic*, or shortest path, in an unusual geometry called the *hyperbolic upper-half plane*. The factor of $1/y$ in the integrand means that planar distances count for more towards the boundary of the upper-half plane (i.e. the x-axis) because $1/y$ gets very big when y gets very small. This accounts for the warped shape of the "straight lines" in this geometry (which are now segments of circles centred on the x-axis) and gives rise to extremely pretty pictures like this one due to M. C. Escher.

Question 3.
Show that

$$y(x) = \cos^{-1}(A\cot x) + B$$

is the general solution to the Euler-Lagrange equation for the variational problem associated to the functional

$$\int \sqrt{1 + (y')^2 \sin^2 x} dx.$$

Hint: Use the two substitutions suggested by the solution!

Remark 2. The paths $\gamma(x) = (x, y(x)\cos x, y(x)\sin x)$ are shortest paths ("great circles") on the unit sphere: the functional is just the length functional $\int |\dot{\gamma}(x)|dx$.

Question 4.
Consider a function $x(t)$ describing the position at time t of a particle of mass m sitting in the force field F whose strength is the gradient of a potential $-V(x)$. Find the Euler-Lagrange equation for the functional

$$\int_0^1 \left(\frac{1}{2}m\dot{x}^2 - V(x)\right) dt$$

and show that solutions obey Newton's law of motion $F = ma$. Interpret Beltrami's identity physically in this situation.

Question 5.
Let $\gamma(t) = (t, t^2\cos(\theta(t)), t^2\sin(\theta(t)))$ be a parametric curve in $\mathbf{R}^3$.

(a) Check that $\gamma(t)$ lies on the surface $S = \{y^2 + z^2 = x^4\}$.

The length of this curve is defined to be the integral $\int |\dot{\gamma}(t)|dt$ where $\dot{\gamma}$ denotes the vector whose components are the t-derivatives of the components of γ.

(b) Write out this integral explicitly as a function of t and $\dot{\theta}(t)$.

(c) Show that if θ solves the corresponding Euler-Lagrange equation (i.e. if γ minimises length amongst paths on S) then

$$\theta(t) = \int \frac{C}{t^2}\sqrt{\frac{1+4t^2}{t^4 - C^2}}dt$$

for some constant C.