Methods 3 - Question Sheet 3

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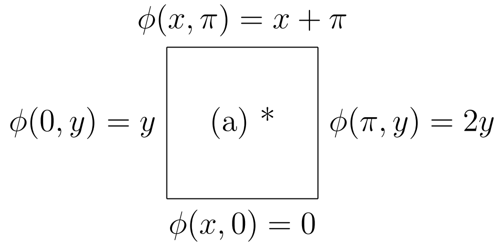
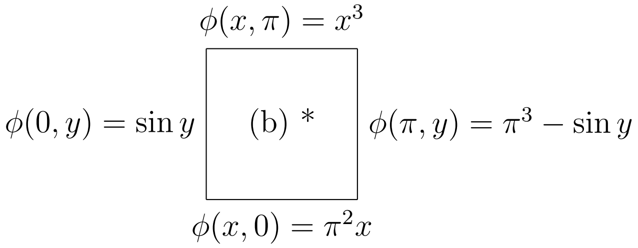
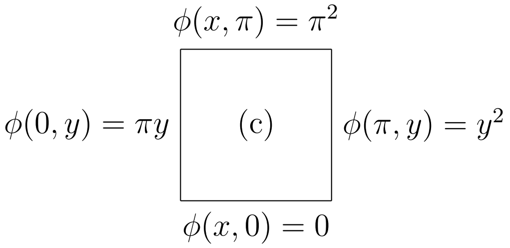
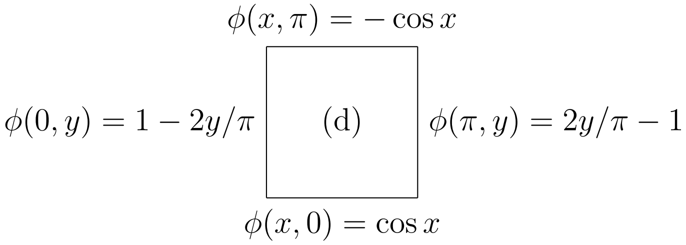
Question 1. (10 marks for * parts)
Solve Laplace's equation for the following boundary value problems. You may use Fourier series you computed in Sheet 1 and standard formulas for general solutions from lectures provided you state them correctly.









Question 2. (5 marks for * part)
Solve the wave equation

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^2 \phi}{\partial t^2}$$

on $x \in [0,1]$ (so $L = 1$) with boundary conditions $\phi(0,t) = \phi(1,t) = 0$ for each of the initial conditions.

(a) * $\phi(x,0) = x - x^2$, $\dfrac{\partial \phi}{\partial t}(x,0) = 0$.

(b) $\phi(x,0) = x - x^3$, $\dfrac{\partial \phi}{\partial t}(x,0) = 0$.

(c) $\phi(x,0) = \cos 2\pi x - 1$, $\dfrac{\partial \phi}{\partial t}(x,0) = 0$.

Question 3. (5 marks)
Suppose that ϕ satisfies the wave equation $\partial_t^2\phi = c^2\partial_x^2\phi$ where c depends discontinuously on x:

$$c(x) = \begin{cases} 1 & \text{if } x < 0 \\ 2 & \text{if } x > 0. \end{cases}$$

We impose the boundary conditions $\phi(-L,t) = \phi(L,t) = 0$ and the conditions that ϕ and $\partial_x\phi$ are continuous at $x = 0$. Let $X(x)T(t)$ be a separated solution; the boundary conditions imply $X(-L) = X(L) = 0$. If $T''/T = -m^2$, prove that m satisfies

$$\tan(mL) = -2\tan(mL/2).$$

Question 4.
Consider the wave equation $\dfrac{\partial^2\phi}{\partial t^2} = \dfrac{\partial^2\phi}{\partial x^2}$ on the interval $[0,1]$.

(a) Find the separated solutions satisfying $\phi(0,t) = 0$, $\dfrac{\partial\phi}{\partial x}(1,t) = 0$.

(b) Find the full solution satisfying the boundary conditions in (a) and the initial conditions

$$\phi(x,0) = \cos(x\pi/2), \quad \frac{\partial\phi}{\partial t}(x,0) = 0.$$

Question 5.
Consider the 2-dimensional heat equation for a temperature distribution $\phi(x,y,t)$

$$\frac{\partial\phi}{\partial t} = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}.$$

(a) By separating variables $\phi(x,y,t) = X(x)Y(y)T(t)$, show that

$$T = e^{Kt}, \quad \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = K\phi$$

(remember this second equation is the Helmholtz equation from Sheet 2).

For a separated solution, the Dirichlet conditions

$$\phi(x,0,t) = \phi(x,1,t) = \phi(0,y,t) = \phi(1,y,t) = 0$$

mean that we require $X(0) = X(1) = Y(0) = Y(1) = 0$. The possible separated solutions to the Helmholtz equation with these boundary conditions were $\sin(px)\sin(qx)$ with $p,q \in \pi\mathbf{Z}$, $K = -(p^2+q^2)$.

(b) What is the solution to the 2-dimensional heat equation with the initial condition $\phi(x,y,0) = x(1-x)\sin(\pi y)$?