Methods 3 - Question Sheet 2

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Question 1. (7 marks)

(a) For each of the following PDEs for $\phi(x,y)$, separate variables ($\phi(x,y) = X(x)Y(y)$) and find the ODEs satisfied by X and Y.

(i) $\dfrac{\partial^2\phi}{\partial x^2} = \dfrac{\partial^2\phi}{\partial y^2} + \dfrac{\partial\phi}{\partial y} + \phi.$
(Telegraph equation, governing lossy wave transmission)

(ii) $\dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} = K\phi$ where K is a constant.
(Helmholtz equation, governing static temperature distributions in 2D)

(b) For equations (i) and (ii) above, solve the ODEs you found.
Be careful to distinguish the values of the separation constant λ where the behaviour of X changes and those where the behaviour of Y changes.

Question 2. (6 marks for * parts)

In each case, solve the heat equation $\dfrac{\partial\phi}{\partial t} = \dfrac{\partial^2\phi}{\partial x^2}$ for a temperature distribution $\phi(x,t)$ on the rod $x \in [0,\pi]$ with the given initial and boundary conditions. You may quote any Fourier series you need from Question Sheet 1:

(a) * $\phi(x,0) = x^3$, $\phi(0,t) = 0$, $\phi(\pi,t) = \pi^3$ (Dirichlet).

(b) * $\phi(x,0) = \cos x$, $\phi(0,t) = 1$, $\phi(\pi,t) = -1$ (Dirichlet).

(c) $\phi(x,0) = x^4 - 2\pi^2x^2$, $\dfrac{\partial\phi}{\partial x}(0,t) = 0$, $\dfrac{\partial\phi}{\partial x}(\pi,t) = 0$ (Neumann).

(d) $\phi(x,0) = \cos x$, $\dfrac{\partial\phi}{\partial x}(0,t) = 0$, $\dfrac{\partial\phi}{\partial x}(\pi,t) = 0$ (Neumann).

(e) $\phi(x,0) = \begin{cases} x & \text{if } x \in \left[0, \frac{\pi}{2}\right] \\ \pi - x & \text{if } x \in \left[\frac{\pi}{2}, \pi\right] \end{cases}$, $\phi(0,t) = 0$, $\phi(\pi,t) = 0$ (Dirichlet).

(f) $\phi(x,0) = e^x$, $\phi(0,t) = 1$, $\phi(\pi,t) = e^\pi$ (Dirichlet).

Question 3. (7 marks)
Suppose $\phi(x, y) = X(x)Y(y)$ is a nontrivial separated solution of the Helmholtz equation $\dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} = K\phi$ satisfying the boundary conditions $X(0) = X(1) = 0$, $Y(0) = Y(1) = 0$.

(a) Show that $\phi(x, y) = C \sin(px) \sin(qy)$ for some $p, q \in \pi\mathbf{Z}$, $C \in \mathbf{R}$.

(b) Deduce that if there is a nontrivial separated solution then $K \in \pi^2\mathbf{Z}$.

(c) If $K = -2\pi^2$, deduce that the only solutions have the form $C \sin(\pi x) \sin(\pi y)$, $C \in \mathbf{R}$, and sketch the graph of such a solution.

(d) How many solutions can you find when $K = -5\pi^2$? When $K = -50\pi^2$?

(e) Why is the space of solutions to the Helmholtz equation (for fixed K) a vector space? What is its dimension?

Question 4.
The Schrödinger equation for the complex probability amplitude $\psi(x, t)$ of a free particle is

$$-i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2}$$

where $\hbar$, m are constants and $i = \sqrt{-1}$.

(a) Separate variables $\psi(x, t) = X(x)T(t)$ and show that $T(t) = e^{iEt/\hbar}$ for some constant E (called the *energy* - in our language it arises as a separation constant).

(b) Assume that $X(0) = X(L)$ (i.e. we are considering a free particle living in the interval $[0, L]$). Show that the energy of a separated solution is *quantised*: it can only take on values $\dfrac{n^2\hbar^2\pi^2}{2mL^2}$.

Question 5.
Let $X(x)T(t)$ be a separated solution to the heat equation and suppose that it satisfies Neumann boundary conditions $X'(0) = X'(L) = 0$. Show that $X = A \cos\left(\frac{n\pi x}{L}\right)$ for some $n \in \mathbf{Z}$ (*Note: $n = 0$ is allowed*).