# Methods 3 - Question Sheet 1

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**Question 1.** (4 marks)
Let $m, n$ be positive integers. Verify the integral identities:

$$\frac{1}{L}\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = \delta_{mn}$$

and

$$\frac{1}{L}\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = 0.$$

**Question 2.** (10 marks for * parts)
For each of the following functions, find its half-range sine series on $[0, \pi]$:

(a) * $f(x) = x^3 - \pi^2 x$.

(b) * $f(x) = \cos x + \dfrac{2x}{\pi} - 1$.

(c) $f(x) = e^x - \dfrac{(e^\pi - 1)x}{\pi} - 1$.

(d) $f(x) = \begin{cases} x & \text{if } x \in \left[0, \frac{\pi}{2}\right] \\ \pi - x & \text{if } x \in \left[\frac{\pi}{2}, \pi\right] \end{cases}.$

(e) $f(x) = \sin x$.

**Question 3.** (6 marks)

(a) Suppose that $F(x)$ is an odd function on $[-\pi, \pi]$. Prove that $\int_{-\pi}^{\pi} F(x)^2 dx = 2\int_0^{\pi} F(x)^2 dx$.

Consider the odd function $F(x)$ on $[-\pi, \pi]$ equal to $x(\pi - x)$ on $[0, \pi]$. The Fourier series of $F$ was found in lectures:

$$F(x) = \frac{8}{\pi}\sum_{n \text{ odd}} \frac{\sin(nx)}{n^3}.$$

(b) Use this to show that

$$T := \frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \cdots = \frac{\pi^6}{960}.$$

(c) Deduce that if

$$S := \frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \cdots$$

then $S = T + \frac{S}{64}$ and hence calculate $S$.

**Question 4.**

Find the half-range cosine series of $f(x) = x^4 - 2\pi^2 x^2$ on $[0, \pi]$. By considering $f(0)$, show that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} = \frac{7\pi^4}{720}$$

**Question 5.**

Imagine there were a function $\delta(x)$ with the property that

$$\int_{-1}^{1} \delta(x) g(x) dx = g(0)$$

for any function $g$. What would the Fourier series of $\delta$ be? What might the graph of the function $\delta$ look like? Let $\delta_N$ denote the approximation to $\delta$ obtained by summing the first $N$ terms of its Fourier series. Show that $\delta_N(x) = \frac{\sin((N+1/2)\pi x)}{2\sin(\pi x/2)}$. Use a computer to plot $\delta_N$ for some small values of $N$. As $N$ increases, does the plot start to look like the graph you imagined?

*$\delta$ is called the Dirac delta function; it is not really a function, but fits into the theory of "distributions". The sequence of truncations is called the Dirichlet kernel; analysis of the integral $\int \delta_N(x) F(x+y) dx$ is involved in proving that the Fourier series of $F$ converges in the $L^2$-sense to $F$.*