Methods 3 - Question Sheet 0

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This is intended as a revision sheet: everything on it was covered at A-level or in Methods 1 and 2. I encourage you to look through it and make sure you remember how to do everything before it comes up in lectures.

Question 1. (Will be useful for Fourier series)
Integrate the following functions (in each case n is an integer).

1. $\int_0^\pi x(x-\pi)\sin(nx)dx$.
2. $\int_0^1 (\cos(2\pi x)-1)\sin(n\pi x)dx$.
3. $\int_0^\pi x\sin(nx)dx$.
4. $\int_0^\pi x^3\sin(nx)dx$.
5. $\int_0^\pi e^x\sin(nx)dx$.

Question 2. (Will be useful for separation of variables)
What is the general solution of the following second order ODEs? In each case, k is a constant.

(a) $y'' = -k^2 y$,

(b) $y'' = k^2 y$,

(c) $y'' + y' + k^2 y = 0$,

(d) $y'' = y + x$.

Question 3. (Will be useful for solving Euler-Lagrange equations)
Do the following integrals (C is a constant):

(a) $\int \frac{du}{\sqrt{u-1}}$,

(b) $\int \frac{du}{\sqrt{1-u^2}}$

(c) $\int \frac{dy}{\sqrt{\frac{1}{Cy^2}-1}}$

(d) $\int \frac{dy}{\sqrt{\frac{1}{-Cy}-1}}$ (Note: This makes sense as long as $-1 < Cy < 0$).

(e) $\int \frac{dt}{\sin t\sqrt{\sin^2 t - C^2}}$ (*Hint: Make a substitution* $u = \cot t$ *to put the integrand into a more familiar form.*).

Question 4. (Will be useful for constrained Euler-Lagrange)
Find the critical points of the function

$$F(x, y, z) = x^2 + 3y^2 + z^2$$

subject to the constraints:

(a) $x^2 + y^2 - z^2 = 2$.

(b) $x^2 + y^2 - z = 2$, $x^2 - y^2 - z = 3$.

Hint: For (b) use two Lagrange multipliers, one for each constraint.

The next two questions focus on the chain rule; make sure you are using the version of the chain rule from Methods 2, valid for functions of several variables, rather than the usual one-variable chain rule!

Question 5. (Will be useful for Euler-Lagrange and method of characteristics)
Let $(u(x, y), v(x, y))$ be a vector-valued function of two variables and let $(x(t), y(t))$ be a path in the plane. If we consider the values of (u, v) along the path, $(u(x(t), y(t)), v(x(t), y(t)))$, what does the chain rule tell us that $\frac{du}{dt}$ and $\frac{dv}{dt}$ are? Show that this is equivalent to the matrix identity:

$$\begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial t} & \frac{\partial v}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix}$$

Find the derivative of $F(x, y) = \sin(xy^2)$ restricted to the path $(x(t), y(t)) = (1 + t, 1 - t)$ at the point $t = 0$:

(a) directly and

(b) using the chain rule.

Question 6. (D'Alembert's method)
Let c be a real number and let $\phi(x, t)$ be a function on $\mathbf{R}^2$. Define u and v by $u = x + ct$ and $v = x - ct$. Find an expression for x and t in terms of u and v and hence (or otherwise) show that

$$\frac{\partial^2 \phi}{\partial u \partial v} = \frac{1}{4}\left(\frac{\partial^2 \phi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2}\right).$$

If ϕ is a solution to $\frac{\partial^2 \phi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2}$, deduce that there exist functions F and G such that $\phi(x, t) = F(x + ct) + G(x - ct)$.