Sheet 7: Hyperbolic geometry

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The final mark out of 10 (which counts towards your grade) will be calculated as your mark on Q1 plus your mark on Q2 plus your best mark from Q3–5. I will also award stars: silver for a total mark of 12 or more on any four questions, gold for a total mark of 15 or more on all questions.

Question 1. (2 marks)
Suppose that $a, b, c, d \in \mathbf{R}$ and $z \in \mathbf{C}$. Find the imaginary part of $\frac{az+b}{cz+d}$.

Question 2. (5 marks)

(a) Find the subgroup of $PSL(2, \mathbf{R})$ consisting of Möbius transformations which fix the point i in the upper half-plane.

(b) Show that the subgroup you found in (a) is isomorphic to $SO(2)$, the group of rotations in 2-dimensional space.

(c) If g is a Möbius transformation with fixed point set P and h is another Möbius transformation, what is the fixed point set of hgh^{-1}?

(d) Let $t_b(z) = z + b$ and consider the group $\{t_b \in PSL(2, \mathbf{R}) \ : \ b \in \mathbf{R}\}$ be the group of translations of the upper half-plane; these all have precisely one fixed point (∞). Conjugate t_b by $h(z) = -1/z$ to get a subgroup of isometries of the hyperbolic upper half-plane which fix 0. Show that, as b varies, the orbit (under the action of this subgroup) of a point ri on the imaginary axis is a Euclidean circle centred at $ri/2$ with radius $r/2$. *[Such a circle is called a horocycle.]*

Question 3. (3 marks)
Consider the upper half-plane model of hyperbolic 2-space. Let C_1, C_2 and C_3 be three hyperbolic lines. We say that C_3 is a common perpendicular if C_3 intersects C_1 and C_2 orthogonally at points Q and R.

(a) Assume there exists a common perpendicular for C_1 and C_2. By considering the hyperbolic triangle PQR that would be formed, prove that C_1 and C_2 cannot intersect at a third point P in the upper half-plane.

(b) When C_1 and C_2 are ultraparallel show that there is always a common perpendicular and that this is unique.

Question 4. (3 marks)
Consider the upper half-plane model of hyperbolic space. Let I denote the imaginary axis, so that $r_I(z) = -\bar{z}$ defines the reflection in I. Let C denote the upper unit semicircle $C = \{z \in \mathbf{C} \; : \; |z| = 1, \; \mathrm{Im}(z) > 0\}$. By conjugating r_I by an appropriate Möbius transformation, find the formula for the reflection r_C in the hyperbolic line C. Now let D be the upper semicircle centred at $a \in \mathbf{R}$ with radius r. By conjugating r_C by an appropriate Möbius transformation, deduce that the reflection r_D in the hyperbolic line D is given by $r_D(z) = a + \frac{r^2}{\bar{z}-a}$.

Question 5. (3 marks)
Find the area of a convex hyperbolic n-gon with interior angles $\alpha_1, \ldots, \alpha_n$. Prove that for every $0 < \alpha < \left(1 - \frac{2}{n}\right)\pi$ there is a regular convex hyperbolic n-gon with interior angles equal to α. Sketch this n-gon in both the disc and upper-half plane models. (The disc model is probably easier!)