Sheet 6: More Möbius maps

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The final mark out of 10 (which counts towards your grade) will be calculated as your mark on Q1 plus your mark on Q2 plus your best mark from Q3–5. I will also award stars: silver for a total mark of 12 or more on any four questions, gold for a total mark of 15 or more on all questions.

Question 1. (2 marks)
Suppose that z_1, z_2, z_3, z_4 are distinct points in $\mathbf{C} \cup \{\infty\}$. For which $w \in \mathbf{C} \cup \{\infty\}$ is it impossible that $[z_1, z_2; z_3, z_4] = w$?

Question 2. (5 marks)
Let $S\colon S^2 \to \mathbf{C} \cup \{\infty\}$ denote the stereographic projection and let $x = (x_1, x_2, x_3)$ be a point on S^2.

(a) Show that $|S(x)| \leq 1$ if and only if $x_3 \leq 0$ (so that the southern hemisphere projects stereographically to the unit disc).

(b) What is the stereographic projection of the spherical triangle with vertices $(1,0,0)$, $(0,1,0)$, $(0,0,1)$?

(c) Suppose that $z = S(x_1, x_2, x_3)$ and let θ be the angle between $(0,0,1)$ and (x_1, x_2, x_3). Show that $|z| = \tan(\theta/2)$. [This is the geometric origin of the famous "$tan(x/2)$ substitution".]

(d) Prove that four points in $\mathbf{C} \cup \{\infty\}$ lie on a circle if and only if their cross-ratio is real.

(e) Deduce that if A, B, C, D are points on a circle (cyclically ordered) then the internal angles in the quadrilateral $ABCD$ at B and at D add up to π and that the angle between AB and AC equals the angle between DB and DC.

[Hint: Consider the arguments of the cross-ratios $[D, B; A, C]$ and $[A, B; C, D]$.]

Question 3. (3 marks)
Let $u, v \in \mathbf{C}$ and consider the points $\pi(u), \pi(v) \in S^2 \subset \mathbf{R}^3$ where $\pi\colon \mathbf{C} \to S^2$ is the inverse of stereographic projection. Prove that the Euclidean distance $|\pi(u) - \pi(v)|$ in $\mathbf{R}^3$ is

$$|\pi(u) - \pi(v)| = \frac{2|u-v|}{\sqrt{(1+|u|^2)(1+|v|^2)}}.$$

[*Courage* - this will probably get worse before it gets better.]

Question 4. (3 marks) Consider a Möbius map of the form $Tz = \frac{az+b}{-\bar{b}z+\bar{a}}$ with $|a|^2 + |b|^2 = 1$. Show that if $u, v \in \mathbf{C}$ and $\pi\colon \mathbf{C} \cup \{\infty\} \to S^2$ is the inverse of stereographic projection then $|\pi(Tu) - \pi(Tv)| = |\pi(u) - \pi(v)|$. (You may assume the expression for $|\pi(u) - \pi(v)|$ from the previous question).

Question 5. (3 marks)
By considering the Möbius transformation $z \mapsto 1 - z$ (and its action on $0, 1, \infty$), show that, if A, B, C, D are four points in $\mathbf{C}$, then $[D, B; A, C] = 1 - [C, B; A, D]$. Deduce that if A, B, C, D lie ordered cyclically on a circle then

$$|A-B||C-D| + |D-A||B-C| = |D-B||A-C|.$$

Deduce that, in a regular pentagon $ABCDE$, the ratio of the length of a diagonal to the length of a side is $\frac{1+\sqrt{5}}{2}$ (i.e. the golden ratio).

[Hint: You may use the results in Q.2(e) to prove, for example, that $\frac{(D-A)(B-C)}{(D-C)(B-A)} = -\frac{|D-A||B-C|}{|D-C||B-A|}$ or $\frac{(D-B)(A-C)}{(D-C)(A-B)} = \frac{|D-B||A-C|}{|D-C||A-B|}$.]