Sheet 5: Möbius maps

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The final mark out of 10 (which counts towards your grade) will be calculated as your mark on Q1 plus your mark on Q2 plus your best mark from Q3–5. I will also award stars: silver for a total mark of 12 or more on any four questions, gold for a total mark of 15 or more on all questions.

Question 1. (2 marks)
Suppose that $x = (x_1, x_2, x_3)$ and $-x = (-x_1, -x_2, -x_3)$ are antipodal points on the unit sphere (so $x_1^2 + x_2^2 + x_3^2 = 1$). Let $S\colon S^2 \to \mathbf{C} \cup \{\infty\}$ denote the stereographic projection. Prove that $S(-x) = -\frac{1}{\overline{S(x)}}$.

Question 2. (5 marks)

(a) Prove that the Möbius group is generated by the Möbius maps $t_b z = z + b$ ($b \in \mathbf{C}$), $h_\lambda z = \lambda z$ ($\lambda \in \mathbf{C} \setminus \{0\}$) and $z \mapsto 1/z$.

(b) Prove that the subgroup of Möbius transformations $\frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbf{R}$ and $ad - bc = 1$ is generated by the Möbius maps t_b, ($b \in \mathbf{R}$), h_λ, ($\lambda \in \mathbf{R}$, $\lambda > 0$) and $z \mapsto -1/z$.

(c) What are the fixed points ($z = Tz$) of the Möbius transformation $Tz = \frac{z\cos\theta - \sin\theta}{z\sin\theta + \cos\theta}$?

(d) Let T be the Möbius transformation from (d). Describe the rotation $\pi \circ T \circ S$ on S^2 by giving its axis and the angle of rotation.

Question 3. (3 marks)
Draw an approximate map of the world under stereographic projection.

Question 4. (3 marks) Let $U, V \subset \mathbf{C}$ be open sets and suppose that $f\colon U \to V$ is a *holomorphic map*, in other words for all $z \in U$ there exists a complex number $f'(z)$ such that

$$f(z+w) = f(z) + f'(z)w + \mathcal{O}(w^2)$$

(where $\mathcal{O}(w^2)$ means higher order terms in w). Suppose that $z_0 \in U$ is a point for which $f'(z_0) \neq 0$. If $\gamma(t)$ is a curve in $\mathbf{C}$, let $\dot{\gamma}(0)$ denote the vector $\frac{d}{dt}\big|_{t=0}(\gamma(t))$. Let $\gamma_1(t)$ and $\gamma_2(t)$ be curves with $\gamma_1(0) = \gamma_2(0) = z_0$ and suppose that the angle between the vectors $\dot{\gamma}_1(0)$ and $\dot{\gamma}_2(0)$ at z_0 is θ. Let $\delta_n(t) = f(\gamma_n(t))$, $n = 1, 2$. Prove that $\dot{\delta}_1(0)$ and $\dot{\delta}_2(0)$ meet at an angle θ at the point $f(z_0)$.

[This is a precise way of saying that holomorphic maps are conformal wherever their derivatives are nonvanishing. Note that Möbius maps are holomorphic.]

Question 5. (3 marks)
Let T be a Möbius transformation $T(z) = \frac{az+b}{cz+d}$. If $c \neq 0$, show that T has either one or two fixed points in $\mathbf{C} \cup \{\infty\}$ and that it has precisely one fixed point if and only if $(a+d)^2 - 4(ad-bc) = 0$. If $c = 0$ show that T has either two fixed points (if $a \neq d$), or one fixed point (if $a = d$ and $b \neq 0$), or else T is the identity.

Find the fixed points of the following Möbius maps:

(a) $Tz = 1/z$,

(b) $Tz = e^{i\theta}z$ (for $\theta \in (0, 2\pi)$),

(c) $Tz = z + 1$.