Sheet 4: Spherical geometry

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I will mark all questions and get a total out of 16. Stars will be awarded: silver for marks of 12 or more, gold for marks of 15 or more. The final mark (which counts towards your grade) will be calculated as Q1 plus Q2 plus your best solution from Q3–5.

Question 1. (2 marks)
Let p and q be two points on the unit circle so that the angle between them is θ. Find the Euclidean length of the straight line pq in $\mathbf{R}^2$.

Question 2. (5 marks)
The moral of this question is that very small spherical triangles look a lot like Euclidean triangles. This is why, although we live on the surface of a sphere (the Earth), ordinary Euclidean geometry appears to apply to the triangles we draw on small scales.

(a) Suppose that Δ is a spherical triangle with side lengths a, b, c and that the angle γ opposite the side of length c is a right-angle. Suppose that a, b, c are very small. By taking the Taylor expansion of $\cos$ in the spherical Pythagorean theorem, prove that

$$a^2 + b^2 \approx c^2.$$

(b) Fix a point $x \in S^2$ and consider the *spherical circle* of points $\{y \in S^2 \ : \ d(x, y) = r\}$. Prove that this is a Euclidean circle of radius $\sin r$ in $\mathbf{R}^3$ and deduce that a spherical circle of spherical radius r has circumference $2\pi \sin r$.

(c) Show that the answer to part (b) approximates the usual formula for the circumference of a Euclidean circle when r is very small.

(d) By integrating the formula for the circumference, find the area bounded by a spherical circle of radius r.

(e) Show that the formula for the area bounded by a spherical circle approximates the usual formula for the area bounded by a Euclidean circle when the spherical radius r is very small.

Question 3. (3 marks)
Suppose that the sphere is subdivided into a collection of convex spherical polygons $P_1, \dots, P_F$. Let

- V denote the number of vertices in the subdivision,
- E denote the number of edges in the subdivision,
- F denote the number of polygons in the subdivision,
- n_i denote the number of edges of P_i,
- α_i denote the sum of the internal angles of P_i.

Prove that

(a) $2\pi V = \sum_{i=1}^{F} \alpha_i$.

(b) $2E = \sum_{i=1}^{F} n_i$.

(c) $V - E + F = 2$. [Hint: Use the Gauss-Bonnet theorem for spherical polygons.]

Question 4. (3 marks) Prove the following statements:

(a) Any isometry of S^2 can be written as a product of at most three reflections in spherical lines.

(b) If a spherical triangle has internal angles π/p, π/q, π/r with $2 \leq p \leq q \leq r$ then (p, q, r) is one of:

- $(2, 2, k)$ for any $k \geq 2$,
- $(2, 3, k)$ for any $2 \leq k \leq 5$.

[Hint: A triangle must have positive area.]

(c) For each of the possible (p, q, r) in part (b), there exists a spherical triangle with these internal angles.

Question 5. (3 marks)
If A, B, C is a spherical triangle, define the *polar triangle* A', B', C' whose vertices are the unit vectors $A' = \frac{B \times C}{\sin a}, B' = \frac{C \times A}{\sin b}, C' = \frac{A \times B}{\sin c}$. Prove that the side lengths of A', B', C' are $\pi - \alpha, \pi - \beta, \pi - \gamma$ and the angles are $\pi - a, \pi - b, \pi - c$.

[Hint: In the notation from lectures, $A' = -n_a$, etc.]

Deduce that

$$\sin \alpha \sin \beta \cos c = \cos \gamma + \cos \alpha \cos \beta.$$

Deduce that the side lengths of a spherical triangle are determined by its internal angles.