Sheet 3: Quaternions and rotations

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The final mark out of 10 (which counts towards your grade) will be calculated as your mark on Q1 plus your mark on Q2 plus your best mark from Q3–5. I will also award stars: silver for a total mark of 12 or more on any four questions, gold for a total mark of 15 or more on all questions.

Question 1. (2 marks)
In each case, find unit quaternions q such that the map $\mathrm{Im}(\mathbf{H}) \to \mathrm{Im}(\mathbf{H})$, $x \mapsto qxq^{-1}$, is the specified orthogonal transformation:

(a) $\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

(b) $\begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}$.

Question 2. (5 marks)

(a) Let $q_1 = t_1 + ix_1 + jy_1 + kz_1$ and $q_2 = t_2 + ix_2 + jy_2 + kz_2$. Show that

$$\mathrm{Re}(\bar{q}_1 q_2) = t_1 t_2 + x_1 x_2 + y_1 y_2 + z_1 z_2.$$

(b) Suppose that $t_1 = t_2 = 0$. Show that $q_1 q_2 = -q_1 \cdot q_2 + q_1 \times q_2$, i.e. the real part equals $-(x_1x_2 + y_1y_2 + z_1z_2)$ and the imaginary part equals $i(y_1z_2 - y_2z_1) + j(z_1x_2 - z_2x_2) + k(x_1y_2 - x_2y_1)$.

(c) Show that $A\colon G \times G \to \mathrm{Maps}(\mathbf{H}, \mathbf{H})$, $A(g,h)x = gxh^{-1}$ defines an action of $G \times G$ on $\mathbf{H}$.

(d) Show that $A(g)\colon x \mapsto gxh^{-1}$ is an isometry of $\mathbf{H}$ when $g, h \in G$.

(e) Parts (c) and (d) imply that we have a homomorphism $G \times G \to O(4)$. Show that the kernel of this homomorphism is $\{(1,1), (-1,-1)\}$. [Hint: Consider the effect of the isometry $x \mapsto gxh^{-1}$ on the quaternions $x = 1$, $x = i$, $x = j$]

Question 3. (3 marks)
Suppose that y is a unit quaternion and let $H \subset \mathbf{H}$ be the plane orthogonal to y. Show that the reflection in the plane H is given in terms of quaternionic multiplication by $z \mapsto -y\bar{z}y$.

Question 4. (3 marks) Consider the 2-by-2 complex matrices:

$$\sigma_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Show that $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = -1$ and that $\sigma_a\sigma_b = \sigma_c = -\sigma_b\sigma_a$ where a, b, c is a cyclic permutation of $1, 2, 3$.

Consider the group of 2-by-2 complex matrices

$$SU(2) = \{A \,:\, A^\dagger = A^{-1},\ \det A = 1\} \text{ where } \dagger \text{ denotes conjugate transpose.}$$

Show that if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(2)$ then $c = -\bar{b}$, $d = \bar{a}$ and $|a|^2 + |b|^2 = 1$. If $a = t + ix$ and $b = y + iz$ then show that

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} = t\mathbf{1} + x\sigma_1 + y\sigma_2 + z\sigma_3$$

and deduce that $SU(2)$ is isomorphic to the group of unit quaternions.

Question 5. (3 marks)
We define the *exponential* of a purely imaginary quaternion q by the formula

$$e^q := \sum_{m=0}^{\infty} \frac{1}{m!} q^m.$$

If $q = u\theta$ for $|u| = 1$, prove that $e^q = \cos\theta + u\sin\theta$.

Check that

$$e^{i\pi/2}e^{j\pi/2} \neq e^{(i+j)\pi/2}$$

(i.e. the usual law of logarithms fails).

[This is just the beginning of a much more general story of exponential maps for groups. For more, see the 4th year course **Lie groups and Lie algebras**.]