Sheet 2: Euclidean isometries

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I will mark all questions and get a total out of 16. Stars will be awarded: silver for marks of 12 or more, gold for marks of 15 or more. The final mark (which counts towards your grade) will be calculated as Q1 plus Q2 plus your best solution from Q3–5.

Question 1. (2 marks)
Let $Sx = Ax + b$ and let $Tx = x + c$. Find S^{-1} in the form $A'x + b'$. Prove that $S \circ T \circ S^{-1}$ is a translation.

Question 2. (5 marks)

(a) Show that the map $\mathrm{Isom}(\mathbf{R}^n) \to O(n)$ given by $T \mapsto A$ (where $Tx = Ax + b$) is a homomorphism. What is its kernel?

(b) Is it true that $\mathrm{Isom}(\mathbf{R}^n)$ is isomorphic to $O(n) \times \mathbf{R}^n$ where $O(n)$ is the subgroup of orthogonal transformations and $\mathbf{R}^n$ is the subgroup of translations? [Hint: Consider a commutator.]

(c) Consider the map $\sigma\colon O(n) \to O(n)$, $\sigma(A) = A\det(A)$. Show that σ is a homomorphism. When n is odd, show that the image is $SO(n)$. Using this, prove that if n is odd, $O(n) \cong SO(n) \times \{\pm\mathbf{1}\}$, where $\mathbf{1}$ denotes the identity matrix.

(d) Deduce that if P is a 3-dimensional convex polytope invariant under the map $\tau\colon \mathbf{R}^3 \to \mathbf{R}^3$, $\tau(x) = -x$ then $\mathrm{Sym}(P) = \mathrm{Sym}^+(P) \times \{\pm 1\}$.

(e) Give an example of a P which is τ-invariant, and an example of a P which shows that τ-invariant is a necessary hypothesis in part (d).

Question 3. (3 marks)
Suppose that G is a finite group acting on a set X. For each $g \in G$, let $X^g = \{x \in X \;:\; gx = x\}$ denote the set of fixed points of g. By considering the set $\{(g,x) \in G \times X \;:\; gx = x\}$, show that $\sum_{g\in G} |X^g| = \sum_{x\in X} |\operatorname{Stab}(x)|$. Deduce that there are $\frac{1}{|G|}\sum_{g\in G} |X^g|$ different orbits in total.

There are $3^4 = 81$ different ways of colouring in the faces of a regular tetrahedron using the colours red, white and blue. The symmetry group A_4 of the tetrahedron acts on these colourings and we say that two colourings are congruent if they are related by a rotation. The number of orbits is the number of congruence classes of colouring. Using the first part of the question, and appealing to the geometric description of the elements of A_4 on Sheet 1, prove that there are 15 congruence classes of colouring.

Question 4. (3 marks)

(a) Let R be a rotation of the plane around 0 by an angle θ. Express R as a product of two reflections.

(b) Let ABC be a triangle with internal angles α, β, γ at A, B, C respectively and let e_A, e_B, e_C denote the edges opposite to A, B, C respectively. Let $R(p, \theta)$ denote the rotation around a point p by an angle θ anticlockwise. Using part (a) or otherwise, show that $R(C, 2\gamma)R(B, 2\beta)R(A, 2\alpha) = 1$.

Question 5. (3 marks)
Let $H \subset \mathbf{R}^n$ be a hyperplane. Show that if T is an isometry of $\mathbf{R}^n$ such that $Tx = x$ for all $x \in H$ then T is either the identity or the reflection in H.