Sheet 1: Polytopes and group actions

J. Evans

I will mark all questions and get a total out of 16. Stars will be awarded: silver for marks of 12 or more, gold for marks of 15 or more. The final mark (which counts towards your grade) will be calculated as Q1 plus Q2 plus your best solution from Q3–5.

Question 1. (2 marks)
Let G be a group acting on a set X; let $x \in X$ and $g \in G$. Show $\mathrm{Stab}(gx) = g\,\mathrm{Stab}(x)g^{-1}$.

Question 2. (5 marks)
In each case, find the size of the group using the orbit-stabiliser theorem applied to a suitable action.

(a) S_n, the group of permutations of n objects.

(b) The symmetry groups of the Platonic solids, using their actions on edges (rather than faces or vertices).

(c) The symmetry group of a 4-dimensional cube (how many facets does it have and what shape are they?).

(d) The symmetry group of the 120-cell, the regular convex 4-dimensional polytope with Schäfli symbol $\{5, 3, 3\}$ (120 dodecahedral facets).

(e) The symmetry group of the snub cube and one more archimedean solid (look them up on Wikipedia: there are 13 - pick your favourite).

Question 3. (3 marks)
The alternating group A_4 contains (in addition to the identity) 3 elements of order 2 (conjugate to $(12)(34)$) and 8 elements of order 3 (conjugate to (123)). Geometrically, how do these act by rotations on the tetrahedron?

Question 4. (3 marks)

(a) What is the convex hull of the eight points $(\pm 1, \pm 1, \pm 1)$ in $\mathbf{R}^3$? What is the convex hull of the four points $\{(1,1,1), (-1,-1,1), (-1,1,-1), (1,-1,-1)\} \subset \mathbf{R}^3$?

(b) Show that the convex hull of a finite set of points is convex.

(c) What is the convex hull of $\{-1, 0, 1\} \subset \mathbf{R}$? Let $X \subset \mathbf{R}^n$ be a finite set of points, let $x \in X$ and let $Y = X \setminus \{x\}$. Suppose that $x \in \mathrm{Conv}(Y)$. Show that $\mathrm{Conv}(Y) = \mathrm{Conv}(X)$. Deduce that a convex polytope is the convex hull of its vertices.

Question 5. (3 marks)
By considering the facets and the vertex figure, prove that there are at most six regular convex polytopes in four dimensions. You may use the fact that the dihedral angles (in radians) for the Platonic solids are:

T	O	I	C	D
$\cos^{-1}(1/3)$	$\pi - \cos^{-1}(1/3)$	$\pi - \cos^{-1}(\sqrt{5}/3)$	$\pi/2$	$\pi - \tan^{-1}(2)$

and that the dihedral angles around a ridge must sum to less than 2π radians.