# HYDRODYNAMICS AND INFINITE-DIMENSIONAL RIEMANNIAN GEOMETRY

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These are the notes from a propaganda-talk for the book of Arnold-Khesin called "Topological Methods in Hydrodynamics" [1]. It also draws heavily from Khesin-Wendt "The geometry of infinite-dimensional groups" [4].

## 1. Lie groups and the symplectic geometry of coadjoint orbits

Let $G$ be a (possibly Frèchet) Lie group, $\mathfrak{g}$ its Lie algebra and $\mathfrak{g}^*$ the dual of $\mathfrak{g}$ (the space of momenta). The latter is the space of continuous linear functionals on $\mathfrak{g}$ and as such is not naturally a Lie algebra (in the infinite dimensional case it need not even be a Frèchet space). We will see that the Lie algebra structure on $\mathfrak{g}$ shows up geometrically on $\mathfrak{g}^*$ as a Poisson structure.

**Theorem 1.** *$\mathfrak{g}^*$ admits a Poisson bracket*

$$\{,\} : \mathcal{C}^\infty(\mathfrak{g}^*) \times \mathcal{C}^\infty(\mathfrak{g}^*) \to \mathcal{C}^\infty(\mathfrak{g}^*)$$

*i.e. an $\mathbb{R}$-bilinear, antisymmetric form which satisfies the Jacobi identity and such that $\{f, -\}$ is a derivation for any $f$. Such a derivation defines a vector field on $\mathfrak{g}^*$ called $v_f$, the* Hamiltonian vector field *associated to $f$ and the span of the Hamiltonian vector fields gives an integrable distribution on $\mathfrak{g}^*$ whose integral leaves are the orbits of the coadjoint action of $G$ on $\mathfrak{g}^*$. These inherit a natural symplectic structure.*

In the infinite dimensional case we need to take care. We must restrict attention to *regular* functions $f \in \mathcal{C}^\infty(\mathfrak{g}^*)$ whose derivative $df : \mathfrak{g}^* \to \mathbb{R}$ is in the subspace $\mathfrak{g} \subset \mathfrak{g}^{**}$.

*Proof.* The Poisson bracket is defined by

$$\{f, g\}(m) := \langle m, [df, dg]\rangle$$

where $m \in \mathfrak{g}^*$, $\langle,\rangle$ is the natural pairing $\mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ and $df$ and $dg$ are considered as elements of $\mathfrak{g}$. This makes sense if we restrict to regular functions because then the Poisson bracket is both well-defined and regular.

The Jacobi identity implies integrability of the distribution on $\mathfrak{g}^*$ spanned by Hamiltonian vector fields. To identify the leaves with coadjoint orbits, let $v_h$ be a Hamiltonian vector field on $\mathfrak{g}^*$, $m \in \mathfrak{g}^*$ and $f$ a regular function on $\mathfrak{g}^*$.

$$\begin{aligned} df(v_h) &= v_h(f)_m \\ &= \{h, f\}(m) \\ &= \langle m, [dh, df]\rangle \\ &= \langle m, \mathrm{ad}_{dh} df\rangle \\ &= \langle \mathrm{ad}^*_{dh} m, df\rangle \end{aligned}$$

so that $v_h = \mathrm{ad}^*_{dh}$. Therefore the span of the Hamiltonian vector fields at $m$ is precisely the tangent space to the coadjoint orbit through $m$ at $m$. The Poisson structure gives us a symplectic structure

$$\Omega(v_f, v_g) = \{f, g\}$$

on Hamiltonian vector fields and hence on the leaves of the Poisson structure. □

## 2. Geodesics on groups

2.1. **The Euler-Arnold equation.** We now restrict to finite dimensional groups and consider the following problem. Let $E$ be a positive definite quadratic form on $\mathfrak{g}$. Using left translation we can generate a left-invariant metric (also written $E$) on $G$ which coincides with $E$ on the tangent space at the identity. Write $A$ for the musical isomorphism

$$A : \mathfrak{g} \to \mathfrak{g}^*.$$

We are interested in the geodesic flow of this metric on $TG$. Let $\gamma$ be a geodesic in $TG$. Left translation gives a family of vectors $L^{-1}_{\gamma(t)}\dot{\gamma}(t)$ in $\mathfrak{g}$. Here's how you should think of this:

**Example 1.** *Let $G = SO(3)$. This is the space of configurations of a rigid body in $\mathbb{R}^3$ with fixed centre of mass. The left translation of the velocity vector of a geodesic $\dot{\gamma}(t)$ back to the origin is just "how the body sees its angular velocity".*

We will further obfuscate by passing to the dual space $\mathfrak{g}^*$. This corresponds to "how the body sees its angular momentum".

**Theorem 2.** *If $\gamma$ is a geodesic of this metric then $m(t) = A\dot{\gamma}(t)$ satisfies the Euler-Arnold equation*

$$\frac{dm}{dt} = -ad^*_{A^{-1}m} m$$

Notice that $E$ can be considered as a function on $\mathfrak{g}^*$:

$$E(m) = \langle m, A^{-1}m \rangle$$

whose differential at $m$ is $dE(m) = A^{-1}m$. Therefore the Euler-Arnold equation is just the Hamiltonian flow of this function on the space $\mathfrak{g}^*$.

*Proof.* Let $g$ be a path in $G$. The condition that $g$ is geodesic is the vanishing of the first variation of the energy on paths with fixed endpoints:

$$\begin{aligned} 0 &= \delta \int E_g(\dot{g}, \dot{g}) dt \\ &= \delta \int E(g^{-1}\dot{g}, g^{-1}\dot{g}) dt \\ &= 2 \int E(\delta(g^{-1}\dot{g}), g^{-1}\dot{g}) dt \end{aligned}$$

But

$$\begin{aligned} \delta(g^{-1}) &= -g^{-1}(\delta g) g^{-1} \\ \delta(g^{-1}\dot{g}) &= g^{-1}\delta\dot{g} - g(\delta g) g^{-1}\dot{g} \\ &= (g^{-1}\delta g)^{\cdot} + [g^{-1}\dot{g}, g^{-1}\delta g] \end{aligned}$$

therefore

$$\begin{aligned}0 &= 2\int E((g^{-1}\delta g)^{\cdot}, g^{-1}\dot{g})dt + 2\int E_1([g^{-1}\dot{g}, g^{-1}\delta g], g^{-1}\dot{g})dt \\ &= -2\int E(g^{-1}\delta g, (g^{-1}\dot{g})^{\cdot})dt + 2\int E(\mathrm{ad}_{g^{-1}\dot{g}}(g^{-1}\delta g), g^{-1}\dot{g})dt\end{aligned}$$

Now recall that $E(v, w) = \langle v, Aw\rangle$ and $\dot{A} = 0$ so that

$$0 = -\int \langle g^{-1}\delta g, (Ag^{-1}\dot{g})^{\cdot} + \mathrm{ad}^*_{g^{-1}\dot{g}} Ag^{-1}\dot{g}\rangle dt$$

or (setting $v = g^{-1}\dot{g}$, $m = Av$)

$$\dot{m} = -\mathrm{ad}^*_v m.$$

□

2.2. **Fluid flows.** The interesting part comes when Arnold makes the following leap of faith. Replace your finite-dimensional Lie group by the group of volume-preserving diffeomorphisms of a Riemannian manifold $(M, g)$. Replace left actions everywhere by right actions. The Euler-Arnold equation becomes the Euler equation for an inviscid, incompressible fluid.

- Left action of rotations on $SO(3)$ is rotation of the ambient space (including the velocity vector) and hence preserves the kinetic energy $E$ of a rigid body. This gives a left-invariant metric on $SO(3)$ suitable for Riemannian geometry as above.
- Right action of diffeomorphisms on $M$ is simply by relabelling of the fluid particles. Hence the kinetic energy of the fluid is right-invariant.
- This leads to some sign changes.

The Lie algebra of SDiff$(M)$ is the algebra of divergence-free vector fields

$$\mathrm{div}(v) = \mathfrak{L}_v \mathrm{vol} = 0$$

with the usual Lie bracket. The dual to this is the space of 1-forms modulo exact 1-forms. To see this, define the pairing

$$(\alpha, v) \mapsto \int \iota_v\mu \wedge \alpha$$

where $\alpha$ is a 1-form and $\mu$ is the volume form on $M$. By the formulae of Stokes and Cartan:

$$(df, v) = -\int f d\iota_v\mu = -\int f\mathrm{div}(v)$$

so that exact 1-forms pair trivially with divergence-free vector fields. Let's check that the pairing is otherwise nondegenerate:

- If $v \neq 0$ then $\alpha = \star\iota_v\mu$ pairs non-trivially with $v$.
- If $(\alpha, v) = 0$ for all divergence-free vector fields $v$ then we will construct a function $f$ for which $\alpha = df$. It suffices to show that $\int_\gamma \alpha = 0$ for any piecewise smooth curve $\gamma$, for then we can define $f(q) = \int_\eta \alpha$ where $\eta$ is a curve connecting a fixed point $p$ to $q$. But $\int_\gamma \alpha$ can be approximated by $\int \alpha \wedge \iota_v\mu$ where $v$ is a divergence-free field supported in a $\epsilon$-thick tubular neighbourhood of $\gamma$ with flux 1 across a Seifert surface of $\gamma$.

The coadjoint action of SDiff$(M)$ on $\Omega^1/d\Omega^0$ is just the pullback action. Therefore the coadjoint action of the Lie algebra is given by

$$\mathrm{ad}^*_v u = \mathfrak{L}_v u$$

The Euler-Arnold equation therefore takes the form

$$\dot{[m]} = -\mathfrak{L}_v[m]$$

Here $v$ is the velocity vector field of the fluid and $m$ is $g$-dual to $v$, i.e. $A$ is the musical isomorphism for the Riemannian metric $g$ used to define the kinetic energy of the fluid. Also, $[u]$ denotes the class of $u \in \Omega^1/d\Omega^0$. Removing brackets gives

$$\dot{m} = -\mathfrak{L}_v m - df$$

for some function $f$ (thought of as pressure). This dualises to the usual Euler equations for an inviscid fluid

$$\dot{v} = -\nabla_v v - \nabla f$$

The function $f$ is chosen uniquely to solve the equation

$$\mathrm{div}(\nabla_v v) = -\Delta f$$

since then the vector field $\dot{v}$ is divergence-free and hence tangent to SDiff$(M)$.

2.3. **Why?** What do we gain from this beautiful rephrasing of the Euler equations?

- The change of viewpoint unifies motion of a rigid body and fluid motion into a single framework. This is suggestive of generalisations. For example, [5] the Maxwell-Vlasov equation of plasma physics fits into a similar picture where $G$ is the group of Hamiltonian symplectomorphisms of phase space (and the Hamiltonian function is suitably chosen). As we will see later, taking $G$ to be the Virasoro extension of the diffeomorphism group of the circle gives the KdV equation. Integrability of the KdV equation will be easier to understand when seen in the setting of symplectic manifolds (i.e. coadjoint orbits of the Virasoro group).
- There is a paper of Ebin-Marsden [2] which sets up the theory of the Euler equation rigorously in the framework of Riemannian geometry on a Hilbert-Sobolev (i.e. $L^2_k$) completion of the volume-preserving diffeomorphism group. From this point of view the equations are actually just the 1-parameter flow of a vector field on a Hilbert manifold and therefore well-posed (short-time existence, smooth dependence on initial conditions, etc.). Smooth initial conditions lead to smooth solutions so that the subgroup of honest $\mathcal{C}^\infty$-diffeomorphisms is preserved by the flow.
- One gets insight into stability analysis of fluid flows. Arnold heuristically calculates sectional curvatures of the volume-preserving diffeomorphism group and shows that usually they're negative. Therefore the geodesic deviation equation implies that forecasts are extremely dependent on initial conditions (in a quantifiable way) and hence unreliable.

## 3. The Virasoro group and the KdV equation

3.1. **KdV as an Euler-Arnold equation.** The Virasoro algebra is a central extension of $\mathfrak{vect}(S^1)$

$$1 \to \mathbb{R} \to \mathfrak{vir} \to \mathfrak{vect}(S^1) \to 1$$

Since central extensions of Lie algebras are classified by elements of Lie algebra cohomology $H^2(\mathfrak{g};\mathbb{R})$ and when $\mathfrak{g} = \mathfrak{vect}(S^1)$ it is known that this cohomology group is 1-dimensional [3]. Therefore the Virasoro algebra is the only non-trivial central extension of $\mathfrak{vect}(S^1)$. This is an interesting story for another time. To cut the story short, Lie bracket is given explicitly[1] by

$$[(f\partial_x, a), (g\partial_x, b)] = (-[f\partial_x, g\partial_x], \int_{S^1} f'g''dx)$$

and there is a Frèchet-Lie group with this as its Lie algebra (one can write it explicitly, but we only ever work with the algebra). The dual $\mathfrak{vir}^*$ is the space of pairs $(udx^2, c)$ where $udx^2$ is a quadratic differential and $c$ is a number, because a quadratic differential pairs with a vector field to give a 1-form which can be integrated, so the pairing $\mathfrak{vir}^* \times \mathfrak{vir} \to \mathbb{R}$ is

$$\langle (udx^2, c), (f\partial_x, a)\rangle = \int_{S^1} uf dx + ac$$

We now try to understand the coadjoint action of the Virasoro algebra. This is defined by

$$\langle \mathrm{ad}^*_{(f\partial_x, a)}(udx^2, c), (g\partial_x, b)\rangle = \langle (udx^2, c), \mathrm{ad}_{(f\partial_x, a)}(g\partial_x, b)\rangle$$

so if $\mathrm{ad}^*_{(f\partial_x, a)}(udx^2, c) := (vdx^2, k)$, we get

$$\begin{aligned}\int_{S^1} vgdx + bk &= \langle (udx^2, c), [(f\partial_x, a), (g\partial_x, b)]\rangle \\ &= \langle (udx^2, c), (f'g - g'f, \int_{S^1} f'g''dx)\rangle \\ &= \int_{S^1} (f'u + f'u + fu' + cf''')gdx\end{aligned}$$

so $v = cf''' + 2f'u + fu'$, $k = 0$, i.e.

$$\boxed{\mathrm{ad}^*_{f\partial_x}(udx^2, c) = (cf''' + 2f'u + fu', 0)}$$

**Theorem 3.** *Let $H(f\partial_x, a) = \frac{1}{2}\int_{S^1} f^2dx + a^2$ be a right-invariant quadratic Hamiltonian function on $\mathfrak{vir}$. The Euler-Arnold equation is the KdV equation*

$$\dot{u} + 3uu' + cu''' = 0$$

*Proof.* The 'musical isomorphism' is given by $A(u\partial_x, c) = (udx^2, c)$ so the Euler-Arnold equation is

$$\begin{aligned}\frac{du}{dt} &= -\mathrm{ad}^*_{A^{-1}(udx^2, c)}(udx^2, c) \\ &= -3u'u - cu''' \\ \frac{dc}{dt} &= 0\end{aligned}$$

□

[1]Don't ask me about signs.

3.2. **KdV as an integrable system.** We will now try and understand the sense in which the KdV equation is an integrable system. Recall that a Hamiltonian system on a symplectic $2n$-manifold $M$ is completely integrable if it admits a collection of $n$ functions $H_1, \ldots, H_n$ (*first integrals of the system*) which

- are linearly independent almost everywhere,
- all Poisson commute with the Hamiltonian,
- and all Poisson commute with one another.

The fact that they Poisson commute with the Hamiltonian means they are preserved under the flow. This makes it easier to find solutions because they're confined to live on the (Lagrangian) orbits of the Hamiltonian $\mathbb{R}^n$-action generated by the functions. In fact, since the Hamiltonian is a linear combination of the first integrals (since Lagrangians are minimal dimension coisotropic submanifolds) the Hamiltonian system is linear on each orbit. If $M$ is compact these must be tori by the Arnold-Liouville theorem. In the example we're interested in, however, $M$ is an infinite-dimensional coadjoint orbit of the Virasoro group. So we're looking for an infinite sequence of commuting Hamiltonians.

**Remark 1.** *A function on $\mathfrak{g}^*$ which Poisson-commutes with every function is called a Casimir function. By definition it is constant on the coadjoint orbits. The fact that there might be an infinite number of Casimir functions (as is indeed the case for ideal fluids in even-dimensional manifolds) has no bearing on the integrability or otherwise of the Euler-Arnold flow. First integrals are only interesting if they're non-constant on the coadjoint orbits.*

Let us write $\{,\}$ for the standard Lie-Poisson bracket on $\mathfrak{g}^*$. This is given by

$$\{f, g\}(m) = \langle [df_m, dg_m], m \rangle$$

Given a point $m_0 \in \mathfrak{g}^*$ we can define a corresponding *constant Poisson bracket* by

$$\{f, g\}_0(m) = \langle [df_m, dg_m], m_0 \rangle$$

and linear combinations of these Poisson brackets

$$\lambda\{,\} + \mu\{,\}_0$$

are also Poisson brackets. We have seen that the KdV equation gives the Hamiltonian flow of a quadratic Hamiltonian $H$ on $\mathfrak{vir}^*$ with respect to the standard brackets. We also have

**Lemma 1.** *The KdV equation is the Hamiltonian flow of the function*

$$Q(udx^2, c) = \frac{1}{2} \int_{S^1} (-u^3 + c(u')^2) dx$$

*with respect to the constant Poisson bracket at $(\frac{1}{2}dx^2, 0)$.*

*Proof.* We have

$$\begin{aligned}
\{Q,F\}_0(m) &= \langle [dQ_m, dF_m], (\frac{1}{2}dx^2, 0)\rangle \\
&= \langle \mathrm{ad}_{dQ_m} dF_m, (\frac{1}{2}dx^2, 0)\rangle \\
&= \langle dF_m, \mathrm{ad}^*_{dQ_m}(\frac{1}{2}dx^2, 0)\rangle \\
dQ_m &= \left(\frac{\delta Q}{\delta u}, \frac{\delta Q}{\delta c}\right) \\
\mathrm{ad}^*_{dQ_m}(\frac{1}{2}dx^2, 0) &= \frac{\delta Q'}{\delta u}
\end{aligned}$$

Now

$$\frac{\delta Q}{\delta u} = -\frac{3}{2}u^2 - cu''$$

so the Hamiltonian vector field generated by $Q$ with respect to the Poisson bracket $\{,\}_0$ is

$$\begin{aligned}
\frac{du}{dt} &= -3uu' - cu''' \\
\frac{dc}{dt} &= 0
\end{aligned}$$

which is the KdV equation. □

Now we perform the following amusing construction. We say a vector field has 1-Hamiltonian $A$ if it's the Hamiltonian vector field associated to the function $A$ on $\mathfrak{g}^*$ by the Poisson brackets $\{,\}_0$. We say a vector field has 2-Hamiltonian $B$ if it's the Hamiltonian vector field associated to the function $B$ on $\mathfrak{g}^*$ by the Poisson brackets $\{,\}$. The 2-Hamiltonian $H_2 = H$ generates the KdV vector field which has 1-Hamiltonian $H_3 = Q$. The 2-Hamiltonian $H_3 = Q$ generates a vector field which has 1-Hamiltonian $H_4$. The 2-Hamiltonian $H_4$ generates a vector field which has 1-Hamiltonian $H_5$ and so forth.

**Lemma 2.** *The Hamitonians thus constructed Poisson commute (for either set of brackets).*

*Proof.* Suppose $j > i$.

$$\begin{aligned}
\{H_i, H_j\} &= \{H_{i+1}, H_j\}_0 \\
&= -\{H_j, H_{i+1}\}_0 \\
&= -\{H_{j-1}, H_{i+1}\} \\
&= \{H_{i+1}, H_{j-1}\}
\end{aligned}$$

This eventually reaches either $\{H_k, H_k\} = 0$ or $\{H_k, H_{k+1}\} = \{H_{k+1}, H_{k+1}\}_0 = 0$. Similarly for $\{,\}_0$. □

For this construction to make sense we need to check that the vector field with 2-Hamiltonian $H_k$ is indeed 1-Hamiltonian. This is where the condition that linear combinations of Poisson brackets are still Poisson brackets will come in (an integrability condition). Instead of doing this (which I can't find written down anywhere) we will construct all the Hamiltonian functions at once. To do this we will need the following observation. Form the Poisson bracket $\{,\}_\lambda = \{,\}_0 + \lambda\{,\}$ and pick a Casimir function $h_\lambda$ for this bracket which varies analytically in $\lambda$ with power series

$$h_\lambda = \sum_{i=0}^{\infty} \lambda^i h_i$$

The fact that $h_\lambda$ is a Casimir translates into

$$0 = \{h_\lambda, f\}_\lambda = \{\sum \lambda^i h_i, f\}_0 + \lambda\{\sum \lambda^i h_i, f\}$$

and equating coefficients this gives

$$\{h_0, f\}_0 = 0,\ \{h_i, f\}_0 = -\{h_{i-1}, f\}$$

The only tricky part is in finding a Casimir whose $h_2$ and $h_3$ terms are the ones we wrote down above. The trick is to identify $\mathfrak{vir}^*$ with the space of differential operators $c\partial_x^2 + u - \lambda$ and to observe that the monodromy of a differential operator (an element of $SL(2, \mathbb{R})$) is conjugated by the Virasoro coadjoint action. Its trace is therefore a Casimir analytic in $\lambda$. I won't go into details. Instead, see Khesin-Wendt's beautiful new book [4].

## References

[1] V. Arnold and B. Khesin, *Topological Methods in Hydrodynamics*, Springer-Verlag, Series: Applied Mathematical Sciences v. 125 (1998)

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