EXERCISES FOR *ASPECTS OF YANG-MILLS THEORY*, 1

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Poincaré lemma: ∼∼∼☺ (the bookwork bookworm): Remind yourself what a differential form is, then prove Poincaré's lemma: any closed form on a contractible manifold M is exact. (If you're rusty after a maths-free holiday you can just look this up and consider it revision.) In the case $\mathbb{R}^n$ use the linear nullhomotopy to write down an explicit formula for the antiderivative.

Principal bundles: Recall that a principal $U(1)$-bundle is a manifold L with a free action of $U(1)$ (i.e. no fixed points). The 'base space' of the bundle is the quotient $M = L/U(1)$ and we think of this as spacetime. Show that M is a manifold. There is a projection map $\pi : L \to M$. We call a continuous map $\sigma : M \to L$ a global section if $\pi \circ \sigma = \mathrm{id}_M$ (think of this as assigning to every point of the base a preimage in L). A local section is a continuous map $\sigma : U \to L$ such that $\pi \circ \sigma = \mathrm{id}_U$, where $U \subset M$ is an open set. A principal bundle always admits local sections (e.g. over contractible open sets). Show that it admits a global section if and only if it is trivial.

Gauge transformations: A gauge transformation is a smooth map $g : L \to L$ such that $\pi \circ g = \pi$ (i.e. g preserves fibres) and such that on each fibre g acts as an element of $U(1)$. The group of these is enormous. By thinking about what its Lie algebra would be if it were a Lie group, convince yourself that it is infinite-dimensional.

Show that gauge transformations are just maps $M \to U(1)$. The gauge group (group of gauge transformations) therefore has its components in bijection with the space of homotopy classes of map $M \to S^1$, which is just $H^1(M;\mathbb{Z})$. If this latter fact is unfamiliar, prove it by a) given a map $M \to S^1$ write down a closed 1-form on M whose cohomology class is integral; b) given a closed 1-form on M whose cohomology class is integral, write down an 'antiderivative' in the form of a circle-valued function on M; c) check that the two operations you have defined are mutually inverse.

Connections: We defined a $U(1)$-connection on a principal bundle in Lecture 2 as a ($U(1)$-invariant) field of horizontal spaces in TL which project 1-1 onto TM along π_*. Gauge transformations are diffeomorphisms of L and hence act by pushforward or pullback (i.e. pushforward along the inverse) on connections. Associated to a connection is a covariant derivative which measures how far sections are from being horizontal:

$$\nabla_X \sigma = \alpha(d\sigma)$$

where α is the projection $TL \to TL$ whose kernel is the horizontal distribution. Show that the effect of a gauge transformation on the covariant derivative is

$$(u\nabla)_X \sigma = u_* \nabla_X (u^{-1}\sigma).$$